

Optimal transport in competition with reaction: the Hellinger-Kantorovich distance and geodesic curves

Matthias Liero*, Alexander Mielke*,[†] and Giuseppe Savaré[‡]

September 24, 2015

Abstract

We discuss a new notion of distance on the space of finite and nonnegative measures on $\Omega \subset \mathbb{R}^d$, which we call Hellinger–Kantorovich distance. It can be seen as an inf-convolution of the well-known Kantorovich–Wasserstein distance and the Hellinger–Kakutani distance. The new distance is based on a dynamical formulation given by an Onsager operator that is the sum of a Wasserstein diffusion part and an additional reaction part describing the generation and absorption of mass.

We present a full characterization of the distance and some of its properties. In particular, the distance can be equivalently described by an optimal transport problem on the cone space over the underlying space Ω . We give a construction of geodesic curves and discuss examples and their general properties.

1 Introduction

Starting from the pioneering works [JKO97] and [JKO98], the reinterpretation of certain scalar diffusion equations as so-called Wasserstein gradient flows led to new analytic tools and concepts and gave deeper insight into diffusion problems, see e.g. [Vil09] and [AD*11]. In particular, in connection with suitable convexity properties of the driving functional the abstract theory of gradient flows in metric space developed in [AGS05] provides a sound and comprehensive geometric framework for these evolution equations.

The recent reformulation of classes of reaction-diffusion systems as gradient systems, see [Mie11, Mie13, LiM13], raises the question whether the abstract metric theory can be also developed for this wider class of problems.

Following [Mie11] we understand a gradient system as a triple $(\mathbf{X}, \mathcal{F}, \mathbb{K})$ consisting of a state space \mathbf{X} , a driving functional \mathcal{F} , and an Onsager operator \mathbb{K} . The latter means that \mathbb{K} is a state-dependent, symmetric, and positive semidefinite linear operator. In many cases the Onsager operator \mathbb{K} induces a dissipation distance $D_{\mathbb{K}}$ on the state space \mathbf{X} by minimizing an action functional over all curves connecting two states. Now, the development of a metric theory rests upon the ability to characterize this distance and its properties.

*Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin.

[†]Humboldt-Universität zu Berlin.

[‡]Università di Pavia.

This paper together with the companion paper [LMS15] provides rigorous characterization of such a dissipation distance. It is based on the simple Onsager operator

$$\mathbb{K}_{\alpha,\beta}(u)\xi = -\alpha \operatorname{div}(u\nabla\xi) + \beta u\xi,$$

where $\alpha, \beta \geq 0$ are fixed parameters. Obviously, the Onsager operator $\mathbb{K}_{\alpha,\beta} = \alpha\mathbb{K}_{\text{Wass}} + \beta\mathbb{K}_{\text{c-a}}$ is a sum of a Wasserstein part for diffusion and a creation-annihilation part, which is the simplest case of a reaction term. For the latter part it is not difficult to develop a corresponding analog to the Wasserstein distance \mathbf{W} . For this, we simply note that $\mathbb{K}_{\text{c-a}}$ is the inverse of a metric tensor, such that the formal associated Riemannian structure is given by $v \mapsto \int_{\Omega} v^2 / (\beta u) dx$. Thus, it is easy to see that the Riemannian distance induced by $\mathbb{K}_{0,\beta}$ is a multiple of the Hellinger–Kakutani distance \mathbf{H} , see [Hel09, Kak48] and [Sch96], where also the correct function spaces are discussed. In particular, for measures of the form $\mu_j = f_j dx$ we obtain

$$\mathbf{D}_{0,\beta}(\mu_0, \mu_1) = \frac{2}{\sqrt{\beta}} \mathbf{H}(\mu_0, \mu_1) \text{ with } \mathbf{H}(\mu_0, \mu_1)^2 = \int_{\Omega} \left(\sqrt{f_0} - \sqrt{f_1} \right)^2 dx.$$

Of course, this distance generalizes to the space of finite, nonnegative Borel measures, denoted by $\mathcal{M}(\Omega)$, see Section 2.3. In the following we shall always assume that the domain $\Omega \subset \mathbb{R}^d$ is convex and compact.

We will show that $\mathbb{K}_{\alpha,\beta}$ generates a proper distance $\mathbf{D}_{\alpha,\beta}$ on $\mathcal{M}(\Omega)$, which is formally given in a generalized Benamou–Brenier formulation (see [BeB00]), i.e., by minimizing over all sufficiently smooth curves $s \mapsto \mu(s)$ connecting measures μ_0 and μ_1 , viz.

$$\mathbf{D}_{\alpha,\beta}(\mu_0, \mu_1)^2 = \inf \left\{ \int_0^1 \int_{\Omega} [\alpha |\nabla \xi|^2 + \beta \xi^2] d\mu(s) ds \mid \frac{d}{ds} \mu + \alpha \operatorname{div}(\mu \nabla \xi) = \beta \mu \xi, \mu_0 \overset{\mu}{\rightsquigarrow} \mu_1 \right\}.$$

This characterization also works for reaction-diffusion systems, see [LiM13, Sect. 2(e)].

We call $\mathbf{D}_{\alpha,\beta}$ the *Hellinger–Kantorovich distance*, since it can be understood as an inf-convolution (weighted by α and β) of the Kantorovich–Wasserstein distance \mathbf{W} and the Hellinger distance \mathbf{H} . In particular, geodesic curves for the distance $\mathbf{D}_{\alpha,\beta}$ will optimize the usage of transport against the usage of creation or annihilation. As an outcome of our theory we will find that transport never occurs over distances longer than $\pi\sqrt{\alpha/\beta}$.

For the rest of this introduction we will use the special choice $\alpha = 1$ and $\beta = 4$, which simplifies the notation considerably.

To give a full characterization of $\mathbf{D}_{1,4}$, we go a detour which will highlight the underlying geometry of the distance much better. Motivated by an explicit formula for the distance between two Dirac measures, we introduce the cone space \mathfrak{C}_{Ω} over Ω and define the Hellinger–Kantorovich distance $\mathbf{HK}(\mu_0, \mu_1)$ by lifting the measures μ_j to measures λ_j on the cone and then minimizing the Wasserstein distance $\mathbf{W}_{\mathfrak{C}}$ induced by a suitable cone distance on \mathfrak{C}_{Ω} . It is then easy to show that \mathbf{HK} is indeed a geodesic distance, since $\mathbf{W}_{\mathfrak{C}}$ is a geodesic distance. It is the purpose of Section 4 to show that $\mathbf{D}_{1,4}$ indeed equals \mathbf{HK} .

For this proof, we will rely on a third characterization of the Hellinger–Kantorovich distance, which is given in terms of the entropy-transport functional for calibration measures $\eta \in \mathcal{M}(\Omega \times \Omega)$ given via

$$\mathcal{ET}_{1,4}(\eta; \mu_0, \mu_1) := \int_{\Omega} F_{\mathbf{B}}\left(\frac{d\eta_0}{d\mu_0}\right) d\mu_0 + \int_{\Omega} F_{\mathbf{B}}\left(\frac{d\eta_1}{d\mu_1}\right) d\mu_1 + \int_{\Omega \times \Omega} c_{1,4}(|x_0 - x_1|) d\eta,$$

where $F_B(z) = z \log z - z + 1$, $\eta_i = \Pi_{\#}^i \eta$ denote the usual marginals, and the cost function $c_{1,4}$ is given by

$$c_{1,4}(L) := \begin{cases} -2 \log(\cos L) & \text{for } L < \pi/2, \\ \infty & \text{for } L \geq \pi/2. \end{cases}$$

Since $\mathcal{E}\mathcal{T}_{1,4}(\cdot; \mu_0, \mu_1)$ is convex, it is easy to find minimizers, see [LMS15] for more details and the proof that $\mathbf{HK}(\mu_0, \mu_1)^2 = \min\{\mathcal{E}\mathcal{T}_{1,4}(\eta; \mu_0, \mu_1) \mid \eta \in \mathcal{M}(\Omega \times \Omega)\}$.

To be more specific, we return to the question of computing the distance between to Dirac masses $\mu_j = a_j \delta_{y_j}$ with $y_j \in \Omega$ and $a_j \geq 0$. Looking at connecting curves of the form $\mu(s) = a(s) \delta_{x(s)}$ we can indeed minimize the length of these one-mass point curves (1mp) and find the result

$$D^{\text{1mp}}(a_0 \delta_{y_0}, a_1 \delta_{y_1})^2 = \begin{cases} a_0 + a_1 - 2\sqrt{a_0 a_1} \cos(|y_1 - y_0|) & \text{for } |y_1 - y_0| \leq \pi, \\ a_0 + a_1 + 2\sqrt{a_0 a_1} & \text{for } |y_1 - y_0| \geq \pi. \end{cases} \quad (1.1)$$

In fact, a minimizer exists only for $|y_1 - y_0| < \pi$ where for $|y_1 - y_0| \geq \pi$ the value $D^{\text{1mp}}(a_0 \delta_{y_0}, a_1 \delta_{y_1})$ is an infimum only. However, it will turn out that these curves are only optimal for $|y_1 - y_0| \leq \pi/2$, while for $|y_1 - y_0| > \pi/2$, the two-mass point curve $\mu(s) = (1-s)^2 a_0 \delta_{y_0} + s^2 a_1 \delta_{y_1}$ is shorter, since its squared length is $a_0 + a_1$. Thus, creation and annihilation is better than transport in this case.

Moreover, the formula in (1.1) suggest to introduce a cone distance $d_{\mathfrak{C}}$ on the cone \mathfrak{C}_{Ω} over Ω given by the elements $[x, r]$ for $r > 0$ and the tip \mathfrak{o} which is an identification of $\{[x, 0] \mid x \in \Omega\}$. The cone distance is defined as

$$d_{\mathfrak{C}}([x_0, r_0], [x_1, r_1])^2 := r_0^2 + r_1^2 - 2r_0 r_1 \cos_{\pi}(|x_1 - x_0|) \text{ with } \cos_b a = \cos(\min\{|a|, b\}),$$

see [BBI01, Sect. 3.6.2]. This distance is again a geodesic distance and we can define the associated Wasserstein distance $W_{\mathfrak{C}}$, see Section 3.2.2.

Based on this observation we can now lift measures μ on Ω to measures λ on \mathfrak{C}_{Ω} such that $\mu = \mathfrak{P}\lambda$, where the projection $\mathfrak{P} : \mathcal{M}_2(\mathfrak{C}_{\Omega}) \rightarrow \mathcal{M}(\Omega)$ is defined via

$$\int_{\Omega} \phi(x) d(\mathfrak{P}\lambda)(x) = \int_{\mathfrak{C}_{\Omega}} r^2 \phi(x) d\lambda([x, r]) \quad \text{for all } \phi \in C^0(\Omega).$$

Now, the first definition of the Hellinger–Kantorovich distance is

$$\mathbf{HK}(\mu_0, \mu_1) = \min \left\{ W_{\mathfrak{C}}(\lambda_0, \lambda_1) \mid \mathfrak{P}\lambda_0 = \mu_0, \mathfrak{P}\lambda_1 = \mu_1 \right\}. \quad (1.2)$$

To further analyze this construction, one needs to study the optimality conditions for the lifts, which can be done by exploiting the characterization via $\mathcal{E}\mathcal{T}_{1,4}$, see Theorem 3.6 and Section 3.3.3, where the crucial duality theory is taken from [LMS15].

In Section 4 we finally show the identity $D_{1,4} = \mathbf{HK}$ by a full characterization of all absolutely continuous curves with respect to the distance \mathbf{HK} , see Theorem 4.5. This is done by lifting curves in $\mathcal{M}(\Omega)$ to curves in $\mathcal{M}_2(\mathfrak{C}_{\Omega})$ and using a characterization of absolutely continuous curves with respect to $W_{\mathfrak{C}}$ which can be found in [Lis07]. In Corollary 4.4 we obtain the important result that *all* geodesics curves in $(\mathcal{M}(\Omega), \mathbf{HK})$ are obtained as projections

$$\mu(s) = \mathfrak{P}\lambda(s), \text{ where } \lambda : [0, 1] \rightarrow \mathcal{M}_2(\mathfrak{C}_{\Omega})$$

is a geodesic curve in $(\mathcal{M}_2(\mathfrak{C}_\Omega), \mathbf{W}_\mathfrak{C})$ connecting optimal lifts λ_0 and λ_1 in (1.2). Throughout this work, the notion “geodesic curve”, or shortly “geodesic”, means constant-speed minimal geodesic, viz.

$$\mathbf{HK}(\mu(s), \mu(t)) = |s-t| \mathbf{HK}(\mu(0), \mu(1)) \text{ for all } s, t \in [0, 1].$$

Section 5 is devoted to various examples for geodesic curves, which are obtained by doing optimal lifts to the cone space \mathfrak{C}_Ω and then constructing geodesic curves for the Wasserstein distance $\mathbf{W}_\mathfrak{C}$ and projecting them down. Since the geodesic curves on the cone \mathfrak{C}_Ω are explicit, this provides an explicit formula for geodesic curves $\mu : [0, 1] \rightarrow \mathcal{M}(\Omega)$, as soon as the lifts are specified.

In particular, using this explicit construction we show that the total mass $m(s) = \mu(s)(\Omega)$ along geodesic curves is 2-convex and 2-concave since we have the identity

$$m(s) = (1-s)m(0) + sm(1) - s(1-s)\mathbf{HK}(\mu_0, \mu_1)^2. \quad (1.3)$$

We discuss geodesic Λ -convexity of some functionals, in particular, we show that the linear functional $\mathcal{F}(\mu) = \int_\Omega \Phi(x) d\mu(x)$ is geodesically Λ -convex if and only if the function $[x, r] \mapsto r^2\Phi(x)$ is geodesically Λ -convex in $(\mathfrak{C}_\Omega, \mathbf{d}_\mathfrak{C})$.

It is also worth to note that the unique geodesic connecting μ_1 to the null measure $\mu_0 \equiv 0$, which has the lifts $\alpha\delta_o$ for $\alpha \geq 0$, is done by the unique Hellinger geodesic

$$\mu^H(s) = s^2\mu_1.$$

This simple observation immediately shows that the logarithmic entropy given by $\mathcal{E}(\mu) = \int_\Omega F_B(u(x)) dx$ for $\mu = u dx$ is not geodesically Λ -convex, since

$$\mathcal{E}(s^2\mu) = s^2\mathcal{E}(\mu) + s^2 \log(s^2)\mu(\Omega) + 1-s^2.$$

In Section 5.2 we reconsider our standard example of the geodesic connections of Dirac masses $\mu_j = a_j\delta_{y_j}$. It turns out that in the critical case $|y_0 - y_1| = \pi/2$ there is an infinite dimensional convex set of geodesic curves that can be constructed by showing that there are many optimal lifts to the cone space \mathfrak{C}_Ω .

Section 5.3 provides a generalization of the classical dilation of measures in the Wasserstein case. For the Hellinger–Kantorovich distance there is a similar dilation where the mass inside the ball $\{x \mid |x - y_0| < \pi/2\}$ is radially transported and partly annihilated into the point y_0 while the mass at larger distance is simply annihilated according to the Hellinger distance.

In Section 5.4 we show how the transport of two characteristic functions occurs in the Hellinger–Kantorovich case. While the too distant parts are simply annihilated or created according to the Hellinger metric the parts that are close enough lead to a continuous transition, see Figure 8.

In Section 5.5 we show that the Hellinger–Kantorovich geodesic between two measures μ_0 and μ_1 is unique if one of the two measures is absolutely continuous with respect to the Lebesgue measure. Finally, Section 5.6 shows that \mathbf{HK} is not semiconcave in $\mathcal{M}(\Omega)$ if $\Omega \subset \mathbb{R}^d$ has dimension two or higher, which is in sharp contrast to the Wasserstein distance, see [AGS05, Def. 12.3.1].

This work, together with its companion paper [LMS15], will form the basis of subsequent work where we will explore the metric properties of the space $(\mathcal{M}(\Omega), \mathbf{HK})$ and study gradient systems on this space. In particular, in the spirit of [LiM13], we aim to establish a metric theory for scalar reaction-diffusion equations of the form

$$\dot{u} = -\mathbb{K}_{\alpha,\beta}(u)\delta\mathcal{F}(u) = \operatorname{div}(\alpha u \nabla(\delta\mathcal{F}(u)) - \beta u \delta\mathcal{F}(u),$$

where $\delta\mathcal{F}$ denotes a variational derivative.

Note during final preparation. The earliest parts of the work presented here were first presented at the ERC Workshop on Optimal Transportation and Applications in Pisa in 2012. Since then the authors developed the theory continuously further and presented results at different workshops and seminars. We refer to [LMS15, Sect. A] for some remarks concerning the chronological development. In June 2015 they became aware of the parallel work [KMV15]. Moreover, in mid August 2015 we became aware of [CP*15a, CP*15b]. So far, these independent works are not reflected in the present version of this manuscript.

2 Gradient structures for reaction-diffusion equations

2.1 General philosophy for gradient systems

We call a triple $(\mathbf{X}, \mathcal{F}, \Psi)$ a gradient system in the differentiable sense, if \mathbf{X} is a Banach space containing the states u , if the functional $\mathcal{F} : \mathbf{X} \rightarrow \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ has a Fréchet subdifferential $D\mathcal{F}(u) \in \mathbf{X}^*$ on a suitable subset of \mathbf{X} , and if Ψ is a dissipation potential. The latter means that $\Psi(u, \cdot) : \mathbf{X} \rightarrow [0, \infty]$ is a lower semicontinuous and convex functional with $\Psi(u, 0) = 0$. Denoting by $\Psi^*(u, \cdot) : \mathbf{X}^* \rightarrow [0, \infty]$ the Legendre-Fenchel transform $\Psi^*(u, \xi) = \sup\{\langle \xi, v \rangle - \Psi(u, v) \mid v \in \mathbf{X}\}$, the gradient evolution is given via

$$\dot{u} \in D_\xi \Psi^*(u, -D\mathcal{F}(u)) \quad \text{or equivalently} \quad 0 = D_u \Psi(u, \dot{u}) + D\mathcal{F}(u). \quad (2.1)$$

For simplicity, we assume that the Fréchet subdifferential $D\mathcal{F}$ and the convex subdifferentials $D_u \Psi$ and $D_\xi \Psi^*$ are single-valued, but the set-valued case can be treated similarly by the standard generalizations.

If the map $v \mapsto \Psi(u, v)$ is quadratic, we call the above system a classical gradient system while otherwise we speak of generalized gradient systems. In the classical case we can write

$$\Psi(u, v) = \frac{1}{2} \langle \mathbb{G}(u)v, v \rangle \quad \text{and} \quad \Psi^*(u, \xi) = \frac{1}{2} \langle \xi, \mathbb{K}(u)\xi \rangle,$$

where $\mathbb{G}(u) : \mathbf{X} \rightarrow \mathbf{X}^*$ and $\mathbb{K}(u) : \mathbf{X}^* \rightarrow \mathbf{X}$ are symmetric and positive (semi)definite operators. Since Ψ and Ψ^* form a dual pair we have $\mathbb{G}(u)^{-1} = \mathbb{K}(u)$ and $\mathbb{K}(u)^{-1} = \mathbb{G}(u)$ if we interpret these identities in the sense of quadratic forms. We call \mathbb{G} the Riemannian operator, as it generalizes the Riemannian tensor on finite-dimensional manifolds, while we call \mathbb{K} the Onsager operator because of Onsager's fundamental contributions in justifying gradient systems via his reciprocal relations $\mathbb{K}(u) = \mathbb{K}(u)^*$, cf. [Ons31, Eqn. (1.11)] or [OnM53, Eqs. (2-1)–(2-4)]. Thus, for classical gradient systems the general form (2.1) specializes to

$$\dot{u} = -\mathbb{K}(u)D\mathcal{F}(u) \quad \text{or equivalently} \quad \mathbb{G}(u)\dot{u} = -D\mathcal{F}(u). \quad (2.2)$$

We emphasize that $\mathbb{K}(u)$ maps (a subspace of) \mathbf{X}^* to \mathbf{X} , so generalized thermodynamic driving forces are mapped to rates. Similarly $\mathbb{G}(u)$ maps rates to viscous dissipative forces, which have to balance the potential restoring force $-\mathrm{D}\mathcal{F}(u)$.

Our work follows the same philosophy as in [JKO98, Ott01]: Even though the above gradient structure is only formal, it may generate a new dissipation distance, which can be made rigorous such that finally the gradient structure can be considered as a mathematically sound metric gradient flow as discussed in [AGS05]. For this one introduces the dissipation distance associated with the dissipation potential Ψ , which is defined via

$$D_{\mathbb{K}}(u_0, u_1)^2 := \inf \left\{ \int_0^1 \langle \mathbb{G}(u) \dot{u}, \dot{u} \rangle ds \mid u \in H^1([0, 1]; \mathbf{X}), u(j) = u_j \right\}.$$

2.2 Dissipation distances for reaction-diffusion systems

It was shown in [Mie11] that certain reaction-diffusion systems admit a formal gradient structure, which is given by an Onsager operator \mathbb{K} and a driving functional \mathcal{F} of the form

$$\mathbb{K}(\mathbf{c})\boldsymbol{\xi} = -\operatorname{div}(\mathbb{M}(\mathbf{c})\nabla\boldsymbol{\xi}) + \mathbb{H}(\mathbf{c})\boldsymbol{\xi}, \quad \mathcal{F}(\mathbf{c}) = \int_{\Omega} \sum_{i=1}^I F(c_i) dx,$$

where $\mathbf{c} = (c_i)_{i=1,\dots,I}$ is the vector of non-negative concentrations of the species X_i , $i = 1, \dots, I$, and $\mathbb{M}(\mathbf{c})$ and $\mathbb{H}(\mathbf{c})$ are a symmetric and positive definite mobility tensor and a reaction matrix, respectively. With the diffusion tensor $\mathbb{D}(\mathbf{c}) = \mathbb{M}(\mathbf{c})\mathrm{D}^2\mathcal{F}(\mathbf{c})$ and the reaction term $\mathbf{R}(\mathbf{c}) = \mathbb{H}(\mathbf{c})\mathrm{D}\mathcal{F}(\mathbf{c})$ the generated gradient-flow equation reads

$$\dot{\mathbf{c}} = -\mathbb{K}(\mathbf{c})\mathrm{D}\mathcal{F}(\mathbf{c}) = \operatorname{div}(\mathbb{D}(\mathbf{c})\nabla\mathbf{c}) - \mathbf{R}(\mathbf{c}).$$

As in the theory of the Kantorovich–Wasserstein distance (cf. [Ott98, JKO98, Ott01, Vil09]) the operator $\mathbb{K}(\mathbf{c})$ can be seen as the inverse of a metric tensor $\mathbb{G}(\mathbf{c})$ that gives rise to a geodesic distance between two densities $\mathbf{c}_0, \mathbf{c}_1 \in L^1(\Omega; [0, \infty]^I)$ defined abstractly via

$$D_{\mathbb{K}}(\mathbf{c}_0, \mathbf{c}_1)^2 := \inf \left\{ \int_0^1 \langle \mathbb{K}(\mathbf{c}(t))^{-1} \dot{\mathbf{c}}(t), \dot{\mathbf{c}}(t) \rangle dt \mid \mathbf{c}_0 \overset{\mathbf{c}}{\rightsquigarrow} \mathbf{c}_1 \right\}. \quad (2.3)$$

Here “ $\mathbf{c}_0 \overset{\mathbf{c}}{\rightsquigarrow} \mathbf{c}_1$ ” means that $t \mapsto \mathbf{c}(t)$ is a sufficiently smooth curve with $\mathbf{c}(0) = \mathbf{c}_0$ and $\mathbf{c}(1) = \mathbf{c}_1$.

Since in general the inversion of \mathbb{K} is difficult or even not well-defined, it is better to use the following formulation in terms of the dual variable $\boldsymbol{\xi}(s) = \mathbb{K}(\mathbf{c}(s))\dot{\mathbf{c}}(s)$, namely

$$D_{\mathbb{K}}(\mathbf{c}_0, \mathbf{c}_1)^2 := \inf \left\{ \int_0^1 \langle \boldsymbol{\xi}(t), \mathbb{K}(\mathbf{c}(t))\boldsymbol{\xi}(t) \rangle dt \mid \dot{\mathbf{c}} = \mathbb{K}(\mathbf{c})\boldsymbol{\xi}, \mathbf{c}_0 \overset{\mathbf{c}}{\rightsquigarrow} \mathbf{c}_1 \right\}. \quad (2.4)$$

In our case of reaction-diffusion operators we can make this even more explicit, namely

$$D_{\mathbb{K}}(\mathbf{c}_0, \mathbf{c}_1)^2 := \inf \left\{ \int_0^1 \int_{\Omega} \nabla\boldsymbol{\xi} : \tilde{\mathbb{M}}(\mathbf{c})\nabla\boldsymbol{\xi} + \boldsymbol{\xi} \cdot \mathbb{H}(\mathbf{c})\boldsymbol{\xi} dx dt \mid \right. \\ \left. \dot{\mathbf{c}} = -\operatorname{div}(\tilde{\mathbb{M}}(\mathbf{c})\nabla\boldsymbol{\xi}) + \mathbb{H}(\mathbf{c})\boldsymbol{\xi}, \mathbf{c}_0 \overset{\mathbf{c}}{\rightsquigarrow} \mathbf{c}_1 \right\} \dots \quad (2.5)$$

Finally, we can use the Benamou–Brenier argument [BeB00] to find the following characterization (cf. [LiM13, Sect. 2.5]):

Proposition 2.1 *We have the equivalence*

$$\begin{aligned}
D_{\mathbb{K}}(\mathbf{c}_0, \mathbf{c}_1)^2 &= \inf \left\{ \int_0^1 \int_{\Omega} \Xi : \tilde{\mathbb{M}}(\mathbf{c})\Xi + \boldsymbol{\xi} \cdot \mathbb{H}(\mathbf{c})\boldsymbol{\xi} \, dx \, dt \mid \right. \\
&\quad \left. \dot{\mathbf{c}} = -\operatorname{div}(\tilde{\mathbb{M}}(\mathbf{c})\Xi) + \mathbb{H}(\mathbf{c})\boldsymbol{\xi}, \, \mathbf{c}_0 \overset{c}{\rightsquigarrow} \mathbf{c}_1 \right\} \\
&= \inf \left\{ \int_0^1 \int_{\Omega} \mathbf{P} : \tilde{\mathbb{M}}(\mathbf{c})^{-1}\mathbf{P} + \mathbf{s} \cdot \mathbb{H}(\mathbf{c})^{-1}\mathbf{s} \, dx \, dt \mid \right. \\
&\quad \left. \dot{\mathbf{c}} = -\operatorname{div}(\mathbf{P}) + \mathbf{s}, \, \mathbf{c}_0 \overset{c}{\rightsquigarrow} \mathbf{c}_1 \right\}
\end{aligned} \tag{2.6}$$

where $\boldsymbol{\xi}(t, x) = \mathbb{H}(\mathbf{c}(t, x))^{-1}\mathbf{s}(t, x) \in \mathbb{R}^I$ and $\Xi(t, x) = \tilde{\mathbb{M}}(\mathbf{c}(t, x))^{-1}\mathbf{P}(t, x) \in \mathbb{R}^{I \times d}$.

Proof: Clearly, the right-hand side in (2.6) gives a value that is smaller or equal than that in (2.5), because we have dropped the constraint $\Xi = \nabla \boldsymbol{\xi}$.

To show that the two definitions give the same value, we have to show that for minimizers (do they exist), the constraint $\Xi = \nabla \boldsymbol{\xi}$ is automatically satisfied. For this we use that $\boldsymbol{\xi}$ and Ξ are related by the continuity equation $\dot{\mathbf{c}} = -\operatorname{div}(\tilde{\mathbb{M}}(\mathbf{c})\Xi) + \mathbb{H}(\mathbf{c})\boldsymbol{\xi}$.

Keeping \mathbf{c} fixed (and sufficiently smooth) we can minimize the integral in (2.6) with respect to $\boldsymbol{\xi}$ and Ξ , which is a quadratic functional with an affine constraint. Hence, we can apply the Lagrange multiplier rule to

$$\mathcal{L}(\Xi, \boldsymbol{\xi}, \boldsymbol{\lambda}) = \int_0^1 \int_{\Omega} \Xi : \tilde{\mathbb{M}}(\mathbf{c})\Xi + \boldsymbol{\xi} \cdot \mathbb{H}(\mathbf{c})\boldsymbol{\xi} + \boldsymbol{\lambda} \cdot (\dot{\mathbf{c}} + \operatorname{div}(\tilde{\mathbb{M}}(\mathbf{c})\Xi) - \mathbb{H}(\mathbf{c})\boldsymbol{\xi}) \, dx \, dt$$

to obtain the Euler-Lagrange equations

$$0 = 2\tilde{\mathbb{M}}\Xi - \tilde{\mathbb{M}}\nabla\boldsymbol{\lambda}, \quad 0 = 2\mathbb{H}\boldsymbol{\xi} - \mathbb{H}\boldsymbol{\lambda}, \quad 0 = \dot{\mathbf{c}} + \operatorname{div}(\tilde{\mathbb{M}}(\mathbf{c})\Xi) - \mathbb{H}(\mathbf{c})\boldsymbol{\xi}.$$

From the first two equations we conclude $\Xi = \frac{1}{2}\nabla\boldsymbol{\lambda} = \nabla\boldsymbol{\xi}$, which is the desired result. ■

2.3 Scalar reaction-diffusion equations

On $\Omega \subset \mathbb{R}^d$, which is a bounded and convex domain, we consider scalar equations of the form

$$\dot{u} = \operatorname{div}(a(u)\nabla u) - f(u) \quad \text{in } \Omega, \quad \nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where we assume that f changes sign, such that $f(u)(u-1)$ is positive for $u \in]0, 1[\cup]1, \infty[$. We want to write the above equation as a gradient system $(\mathbf{X}, \mathcal{F}, \mathbb{K})$ with $\mathbf{X} = L^1(\Omega)$,

$$\mathcal{F}(u) = \int_{\Omega} F(u(x)) \, dx, \quad \text{and} \quad \mathbb{K}(u)\boldsymbol{\xi} = -\operatorname{div}(\mu(u)\nabla\boldsymbol{\xi}) + k(u)\boldsymbol{\xi} \quad \text{with } \nabla\boldsymbol{\xi} \cdot \nu = 0,$$

where $F : \mathbb{R} \rightarrow [0, \infty]$ is a strictly convex function with $F(u) = \infty$ for $u < 0$ and $F(1) = 0$. Moreover, we assume $\mu(u), k(u) \geq 0$, such that the dual dissipation potential is the nonnegative quadratic form

$$\Psi^*(u, \boldsymbol{\xi}) = \int_{\Omega} \frac{\mu(u(x))}{2} |\nabla\boldsymbol{\xi}(x)|^2 + \frac{k(u(x))}{2} \boldsymbol{\xi}(x)^2 \, dx.$$

Using $D\mathcal{F}(u) = F'(u(x))$ we obtain

$$\mathbb{K}(u)D\mathcal{F}(u) = -\operatorname{div}(\mu(u)F''(u)\nabla u) + k(u)F'(u).$$

Hence we see that we obtain the above reaction-diffusion equation, if we choose F , μ , and k such that the relations

$$a(u) = \mu(u)F''(u) \quad \text{and} \quad k(u)F'(u) = f(u).$$

There are several canonical choices. Quite often one is interested in the case $a \equiv 1$, which gives rise to the simple semilinear equation $\dot{u} = \Delta u - f(u)$. To realize this one chooses $\mu(u) = 1/F''(u)$. This is particularly interesting in the case of the logarithmic entropy where $F(u) = F_B(u) := u \log u - u + 1$. Then, $\mu(u) = 1/F''(u) = u$ and we obtain the Wasserstein operator $\xi \mapsto -\operatorname{div}(u\nabla \xi)$ for the diffusion part.

For the reaction part one simply chooses $k(u) = f(u)/F'(u)$, which is positive, since $f(u)$ and $F'(u)$ change the sign at $u = 1$. For $F = F_B$ and the equation

$$\dot{u} = \Delta u - \kappa(u^\beta - u^\alpha) \quad \text{with } 0 \leq \alpha < \beta$$

we obtain $k(u) = \kappa(u^\beta - u^\alpha)/\log u = \kappa(\beta - \alpha)\Lambda(u^\alpha, u^\beta)$, where the logarithmic mean is given via $\Lambda(a, b) = (a - b)/(\log a - \log b)$. This equation models the evolution of a single diffusing species undergoing the creation-annihilation reaction

$$\alpha X \xrightleftharpoons[\kappa]{\kappa} \beta X.$$

The simplest example of a reaction-diffusion distance is the Hellinger–Kantorovich distance $D_{\alpha, \beta}$ studied in Section 3 in great detail. It is defined via the scalar Onsager operator

$$\mathbb{K}_{\alpha, \beta}(c)\xi := -\alpha \operatorname{div}(c \nabla \xi) + \beta c \xi, \tag{2.7}$$

where α, β are nonnegative parameters. The special property of this operator is that it is linear in the variable c . This will allow us to do explicit calculations for the corresponding dissipation distance $D_{\alpha, \beta} := D_{\mathbb{K}_{\alpha, \beta}}$. In particular, the associated distance is defined for all pairs of (nonnegative and finite) measures $\mu_0, \mu_1 \in \mathcal{M}(\Omega)$, not just for probability measures $\mathcal{P}(\Omega)$. In fact, we will see that for $\beta > 0$ the geodesic curves connecting to different probability measures will have mass less than one for all arclength parameters $s \in]0, 1[$.

For $\beta = 0$ we obtain the scaled Kantorovich–Wasserstein distance, namely

$$D_{\alpha, 0}(\mu_0, \mu_1) = \begin{cases} \frac{1}{\sqrt{\alpha|\mu_0|}} W\left(\frac{\mu_0}{|\mu_0|}, \frac{\mu_1}{|\mu_1|}\right) & \text{if } |\mu_0| = |\mu_1|, \\ \infty & \text{else.} \end{cases}$$

Here, $|\mu_j| = \mu_j(\Omega)$ is the total mass of the measure. The geodesic curves are given in terms of the classical optimal transport, see [AGS05, Ch. 7].

For $\alpha = 0$ we obtain a scaled version of the Hellinger distance (sometimes also called Hellinger–Kakutani distance), namely

$$D_{0, \beta}(\mu_0, \mu_1) = \frac{2}{\sqrt{\beta}} H(\mu_0, \mu_1) = \frac{2}{\sqrt{\beta}} \left(\int_{\Omega} \left[\left(\frac{d\mu_0}{d\mu_*} \right)^{1/2} - \left(\frac{d\mu_1}{d\mu_*} \right)^{1/2} \right]^2 d\mu_* \right)^{1/2}$$

for a reference measure μ_* with $\mu_i \ll \mu_*$ (e.g. $\mu_* = \mu_0 + \mu_1$), see [Sch96, Theorem 4]. The geodesic curves are given by linear interpolation of the square roots of the densities, i.e.

$$(\mu^H(s))(A) = \int_A \left((1-s) \left(\frac{d\mu_0}{d(\mu_0+\mu_1)} \right)^{1/2} + s \left(\frac{d\mu_1}{d(\mu_0+\mu_1)} \right)^{1/2} \right)^2 d(\mu_0+\mu_1). \quad (2.8)$$

By using the estimate $\mu^H(s) \geq (1-s)^2 \mu_0 + s^2 \mu_1$ and choosing $s \in [0, 1]$ optimally, we obtain the lower estimate $|\mu^H(s)| \geq |\mu_0| |\mu_1| / (|\mu_0| + |\mu_1|)$, i.e. the total mass of the geodesic $\mu^H(s)$ is bounded from below by half of the harmonic mean of the total masses of μ_0 and μ_1 . Moreover, an elementary calculation gives the identity

$$|\mu^H(s)| = (1-s)|\mu_0| + s|\mu_1| - s(1-s)H(\mu_0, \mu_1)^2. \quad (2.9)$$

3 The Hellinger–Kantorovich distance

In this section we discuss the dissipation distance $D_{\alpha,\beta}(\mu_0, \mu_1)$ that is induced by the Onsager operator $\mathbb{K}_{\alpha,\beta}(c)\xi = -\operatorname{div}(\alpha c \nabla \xi) + \beta c \xi$, given for $\mu_0, \mu_1 \in \mathcal{M}(\Omega)$ as in (2.3). Using Proposition 2.1 we can rewrite this formulation in an equivalent form as

$$D_{\alpha,\beta}(\mu_0, \mu_1)^2 = \inf \left\{ \int_0^1 \int_{\Omega} [\alpha |\Xi|^2 + \beta \xi^2] d\mu(s) ds \mid \frac{d}{ds} \mu + \alpha \operatorname{div}(\mu \Xi) = \beta \mu \xi, \mu_0 \overset{\mu}{\rightsquigarrow} \mu_1 \right\} \quad (3.1)$$

with $\Xi : [0, 1] \times \Omega \rightarrow \mathbb{R}^d$ denoting the vector field.

In most of this section we will restrict ourselves without loss of generality to the case $\alpha = 1$ and $\beta = 4$ for simplicity. Occasionally, we will give some of the formulas for general α and β to highlight the dependence on these parameters. Note that we can always use the simple scaling $\mathbb{K}_{\alpha,\beta} = \beta \mathbb{K}_{\alpha/\beta,1}$ giving the general relation $D_{\alpha,\beta}(\mu_0, \mu_1) = D_{\alpha/\beta,1}(\mu_0, \mu_1)/\sqrt{\beta}$. Moreover, the factor $\sqrt{\alpha/\beta}$ can be transformed away by rescaling Ω , i.e. $x \mapsto \sqrt{\alpha/\beta}x$.

Note that for sufficiently regular μ , ξ , and Ξ in (3.1) we obtain by Proposition 2.1 $\Xi = \nabla \xi$ and formal calculation leads to the following system of equations for geodesic curves

$$\dot{\mu} = -\alpha \operatorname{div}(\mu \nabla \xi) + \beta \mu \xi, \quad \dot{\xi} + \frac{\alpha}{2} |\nabla \xi|^2 + \frac{\beta}{2} \xi^2. \quad (3.2)$$

For the case $\beta = 0$ and $\alpha = 1$ this corresponds to [BeB00, Eqn. (37)]. A full justification of this coupled system is given in [LMS15, Sect. 8.6].

3.1 The optimal curves for $D_{\alpha,\beta}$ with one or two mass-points

The striking feature of optimal transport is that for affine mobilities point masses (Dirac measures) are transported as point masses, i.e. the geodesic curve connecting $\mu_0 = \delta_{x_0}$ and $\mu_1 = \delta_{x_1}$ is given by $\mu_s = \delta_{x(s)}$, where $[0, 1] \ni s \mapsto x(s)$ is a geodesic curve in the underlying domain Ω .

Since the Onsager operator $\mathbb{K}_{\alpha,\beta}(c)$ in (2.7) depends only linearly on the state c , we expect a similar behavior. In particular, note that the definition of the distance in (3.1) is well-defined for general curves of measures $\mu(s) \in \mathcal{M}(\Omega)$ if we understand the linear constraint $\frac{d}{ds}\mu + \alpha \operatorname{div}(\mu \Xi) = \beta \mu \xi$ in the distributional sense.

As a first step, it is instructive to study the $\mathbb{K}_{\alpha,\beta}$ -length of curves given by a moving point mass in the form

$$\gamma_{x,a} : [0, 1] \ni s \mapsto \mu(s) = a(s)\delta_{x(s)} \quad \text{with } x(s) \in \Omega \text{ and } a(s) \geq 0.$$

Minimizing the action functional in (3.1) only over curves of this form for given end points $a_i\delta_{x_i}$, $i = 0, 1$, always gives an upper bound for the distance $D_{\alpha,\beta}(a_0\delta_{x_0}, a_1\delta_{x_1})$. Indeed, we show that up to a certain threshold for the Euclidean distance $|x_0 - x_1|$ it will even be the exact distance and the minimizing $\gamma_{x,a}$ is a geodesic curve.

The main point is that we are able to calculate the s -derivative of $\mu(s) = \gamma_{x,a}(s)$ and compare it to the continuity equation. Multiplying the continuity equation with test functions we obtain after integration by parts

$$\frac{d}{ds}\mu_i(s) = -\operatorname{div}(\dot{x}(s)a(s)\delta_{x(s)}) + \dot{a}(s)\delta_{x(s)} = -\operatorname{div}(\dot{x}(s)\mu(s)) + \frac{\dot{a}(s)}{a(s)}\mu(s).$$

Thus, comparing with the continuity equation in the definition of $D_{\alpha,\beta}$ we find the relations

$$\Xi(s, x(s)) = \frac{1}{\alpha}\dot{x}(s) \quad \text{and} \quad \xi(s, x(s)) = \frac{\dot{a}(s)}{\beta a(s)}. \quad (3.3)$$

We may realize the constraint $\Xi = \nabla \xi$ via $\xi(s, y) = \xi(s, x(s)) + \frac{1}{\alpha}\dot{x}(s) \cdot (y - x(s))$.

Having identified the vector and scalar field Ξ and ξ , respectively, we obtain the $\mathbb{K}_{\alpha,\beta}$ -length of the curve $s \mapsto a(s)\delta_{x(s)}$ via

$$\operatorname{Length}_{\alpha,\beta}(\gamma_{x,a})^2 = \int_0^1 \left[\frac{1}{\alpha}|\dot{x}(s)|^2 + \frac{1}{\beta} \left(\frac{\dot{a}(s)}{a(s)} \right)^2 \right] a(s) ds, \quad (3.4)$$

for $\alpha = 0$ and $\beta = 8$ this corresponds to the representation in [Sch96, Thm. 4] for the Hellinger–Kakutani distance. Minimizing this expression for given endpoints of $\gamma_{x,a}$ we find that $x(s)$ travels along a straight line, which reflects the fact that our choice of metric in Ω is the Euclidean one. However, the speed will not be constant. Hence, we introduce functions

$$\begin{aligned} \rho &\in \mathfrak{R}(0, 1) := \{ \rho \in H^1(0, 1) \mid \rho(0) = 0, \rho(1) = 1, \dot{\rho} \geq 0 \} \\ a &\in \mathfrak{A}(a_0, a_1) := \{ a \in H^1(0, 1) \mid a > 0, a(0) = a_0, a(1) = a_1 \} \end{aligned} \quad (3.5)$$

such that we can write $x(s) = (1 - \rho(s))x_0 + \rho(s)x_1$ and have $|\dot{x}(s)| = \dot{\rho}(s)L$ with $L = |x_1 - x_0|$. For the one-mass-point problem we define the function $\mathcal{J}_{\alpha,\beta}^{\text{1mp}} : [0, \infty[\times [0, \infty[^2 \rightarrow [0, \infty[$ via

$$\mathcal{J}_{\alpha,\beta}^{\text{1mp}}(L^2, a_0, a_1) := \inf \left\{ \int_0^1 \frac{L^2}{\alpha} \dot{\rho}(s)^2 a(s) + \frac{\dot{a}(s)^2}{\beta a(s)} ds \mid \rho \in \mathfrak{R}(0, 1), a \in \mathfrak{A}(a_0, a_1) \right\}. \quad (3.6)$$

The functional $\mathcal{J}_{\alpha,\beta}^{\text{1mp}}$ satisfies a scaling identity with respect to the parameter $\alpha, \beta > 0$: For $\theta > 0$ we have

$$\mathcal{J}_{\alpha,\beta}^{\text{1mp}}(L^2, a_0, a_1) = \frac{1}{\theta} \mathcal{J}_{1,\beta/\theta}^{\text{1mp}}\left(\frac{\theta L^2}{\alpha}, a_0, a_1\right). \quad (3.7)$$

Hence, we can restrict ourselves to one particular choice of $\beta/\theta > 0$ such that the general case can be recovered from a rescaling of the Euclidean distance in Ω . In particular, it will prove convenient to choose $\theta = \beta/4$ such that we will consider $\mathcal{J}_{1,4}^{\text{imp}}$ first, which is also the scaling used in [LMS15].

Theorem 3.1 *We have*

$$\begin{aligned} \mathcal{J}_{1,4}^{\text{imp}}(L^2, a_0, a_1) &= a_0 + a_1 - 2\sqrt{a_0 a_1} \cos_\pi(L) \quad \text{with} \\ \cos_\pi(L) &:= \begin{cases} \cos(L) & \text{for } L < \pi, \\ -1 & \text{for } L \geq \pi. \end{cases} \end{aligned} \quad (3.8)$$

The infimum is a minimum for $L < \pi$, and it is attained for

$$\begin{aligned} a(s) &= (1-s)^2 a_0 + s^2 a_1 + 2s(1-s)\sqrt{a_0 a_1} \cos(L), \\ \rho(s) &= \begin{cases} \frac{1}{L} \arctan\left(\frac{s \sin(L) \sqrt{a_1}}{(1-s)\sqrt{a_0} + s \cos(L) \sqrt{a_1}}\right) & \text{if } (1-s)\sqrt{a_0} + s \cos(L) \sqrt{a_1} > 0, \\ \frac{\pi}{2L} & \text{if } (1-s)\sqrt{a_0} + s \cos(L) \sqrt{a_1} = 0, \\ \frac{1}{L} \arctan\left(\frac{s \sin(L) \sqrt{a_1}}{(1-s)\sqrt{a_0} + s \cos(L) \sqrt{a_1}}\right) + \frac{\pi}{L} & \text{otherwise.} \end{cases} \end{aligned} \quad (3.9)$$

For $L \geq \pi$ the minimizing sequences converge to $a(s) = c(s-\theta)^2$ and $\dot{\rho}(s) = \delta_\theta(s)$ for certain $c \geq 0$ and $\theta \in [0, 1]$.

Proof: To study the infimum of $\mathcal{J}_{1,4}^{\text{imp}}$ we transform the system by using $b(s) = \sqrt{a(s)}$. Keeping $L > 0$ fixed we obtain the functional

$$\mathcal{K}_L(b, \rho) = \int_0^1 L^2 b(s)^2 \dot{\rho}(s)^2 + \dot{b}(s)^2 ds.$$

Clearly, the infimum of \mathcal{K}_L gives the infimum in the definition of $\mathcal{J}_{1,4}^{\text{imp}}$. We now consider a minimizing sequence (b_n, ρ_n) and observe that (b_n) must remain bounded in $H^1(0, 1)$. Hence, after choosing a suitable subsequence (not relabeled), we may assume $b_n \rightharpoonup b$ in $H^1(0, 1)$. We distinguish between the following three cases.

Case 1. $\underline{b} := \min\{b(s) \mid s \in [0, 1]\} > 0$: In this case we may further conclude that ρ_n is also bounded in $H^1(0, 1)$. Hence, we can also assume $\rho_n \rightharpoonup \rho$ in $H^1(0, 1)$. Because of the lower semicontinuity of \mathcal{K}_L on $H^1(0, 1)^2$ we conclude that (b, ρ) is the global minimizer of \mathcal{K}_L . This implies that $(a, \rho) = (b^2, \rho)$ is the global minimizer of $\mathcal{J}_{1,4}^{\text{imp}}$, which certainly satisfies the Euler–Lagrange equations

$$\frac{d}{ds}(\dot{\rho}a) = 0, \quad 4(L\dot{\rho})^2 - (\dot{a}/a)^2 - 2\frac{d}{ds}(\dot{a}/a) = 0.$$

From the first equation and $\int_0^1 \dot{\rho} ds = 1$ we obtain

$$\dot{\rho}(s) = H[a]/a(s), \quad \text{where } H[a] = \left(\int_0^1 1/a(s) ds\right)^{-1}$$

denotes the harmonic mean, which satisfies $H[a] \geq \underline{b}^2 > 0$. By inserting this into the second equation we see that all solutions are given in the form

$$a(s) = c_0 + c_1(s-\theta)^2 \quad \text{with } c_0 c_1 = L^2 H[a]^2.$$

Hence, together with the boundary conditions $a_0 = a(0) = c_0 + c_1\theta^2$ and $a_1 = a(1) = c_0 + c_1(1-\theta)^2$ we have three nonlinear equations for the unknowns c_0, c_1 , and θ , which can be solved easily for $L < \pi$ giving a unique solution. For $L \geq \pi$ no solution with positive \underline{b} exists. In particular, since the primitive of the inverse of a strictly positive quadratic function is given in terms of the arctan function we obtain the formulas in (3.9), using also the addition theorem for arctan.

Thus, in the case $\underline{b} > 0$ the global minimizer is the unique, positive critical point (a, ρ) .

Case 2. $\underline{b} = 0$: Since b is continuous, there exists $s_* \in [0, 1]$ with $b(s_*) = 0$. Neglecting the term $L^2 b^2 \dot{\rho}^2$ in the integrand in \mathcal{K}_L we can minimize the remaining quadratic term subject to the boundary conditions $b(0) = \sqrt{a_0}$, $b(1) = \sqrt{a_1}$, and $b(s_*) = 0$. This leads to a minimizer that is piecewise affine and gives the lower bound

$$\mathcal{K}_L(b, \rho) \geq \frac{a_0}{s_*} + \frac{a_1}{1-s_*} \geq (\sqrt{a_0} + \sqrt{a_1})^2, \quad (3.10)$$

where the last estimate follows from minimization in s_* .

It is now easy to see that the value $(\sqrt{a_0} + \sqrt{a_1})^2$ is indeed the infimum, since it can be obtained as a limit of a minimizing sequence. For this take piecewise affine functions (b_n, ρ_n) satisfying $(b_n(s), \dot{\rho}_n(s)) = (0, n)$ for $s \in [s_n, s_n + 1/n]$ with $s_n \rightarrow s_0$, where s_0 is the optimal s_* in (3.10). On $[0, s_n]$ we take $(\dot{b}_n(s), \rho(s)) = (-\sqrt{a_0}/s_n, 0)$ and similarly on $[s_n + 1/n, 1]$.

General case: Since the infimum obtained in case 2 is strictly larger than that in case 1 (because of $L < \pi$), we see that the two cases exclude each other. If $L < \pi$ then case 1 occurs while for $L \geq \pi$ case 2 sets in. Hence, the theorem is established. ■

Although the stationary states in Theorem 3.8 may be the global minimizers, they are not always the geodesic curves with respect to $D_{1,4}$. To see this we consider the pure reaction case and define the curve

$$\widehat{\gamma}_{\bar{a}_0, \bar{a}_1}(s) = \bar{a}_0(s)\delta_{x_0} + \bar{a}_1(s)\delta_{x_1}$$

also connecting the measures $\mu_j = a_j\delta_{x_j}$ if $\bar{a}_j(j) = a_j$ and $\bar{a}_j(1-j) = 0$ for $j = 0, 1$. If $\bar{a}_0(s), \bar{a}_1(s) > 0$, this curve consists of two separated mass points that do not move. As in the previous case of the moving mass point we can compute the solutions of the continuity equation to obtain

$$\Xi \equiv 0 \quad \text{and} \quad \xi(s, x_j) = \frac{\dot{\bar{a}}_j(s)}{4\bar{a}_j(s)}.$$

The squared length of these curves is given by $\frac{1}{4} \sum_j \int_0^1 \dot{\bar{a}}_j^2 / \bar{a}_j \, ds$ and the optimal choice for \bar{a}_j is $\bar{a}_0(s) = a_0(1-s)^2$ and $\bar{a}_1(s) = a_1 s^2$ giving the minimal squared length

$$\text{Length}_{1,4}(\widehat{\gamma}_{\text{opt}})^2 = a_0 + a_1 \geq D_{1,4}(a_0\delta_{x_0}, a_1\delta_{x_1}).$$

We see that this result is less than $\mathcal{J}_{1,4}^{\text{imp}}(|x_1 - x_0|^2, a_0, a_0)$ for $\pi/2 < |x_1 - x_0| < \pi$. In fact, we will show later that the last estimate is sharp if and only if $|x_1 - x_0| \geq \pi/2$.

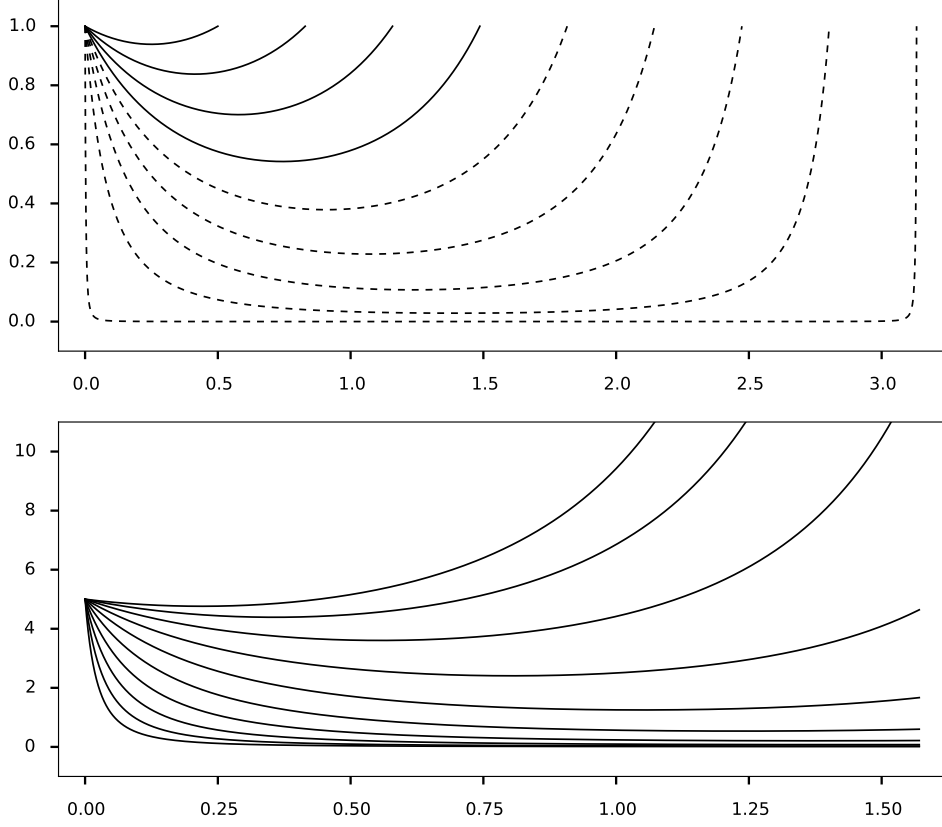


Figure 1: *Top:* The curves $s \mapsto (L \int_0^s \dot{\rho}(\tau) d\tau, a(s))$ for different values of $0 < L < \pi$. Solid curves are true geodesics, while dashed curves are shortest “one-mass-point paths” but not geodesic curves. *Bottom:* Curves for $L = \pi/2$ and different mass ratios a_0/a_1 .

To highlight the dependencies on α and β in $\mathcal{J}_{\alpha,\beta}^{\text{1mp}}$ we can use the scaling (3.7) for all $\alpha, \beta > 0$. To include the limit cases of the Hellinger distance (i.e. $\alpha = 0$) and the Kantorovich–Wasserstein distance (i.e. $\beta = 0$) we define the functions $\mathfrak{S}_{\alpha,\beta}$ via

$$\mathfrak{S}_{\alpha,\beta}(L^2, b_0, b_1) := \begin{cases} \frac{4}{\beta} \left(b_0^2 + b_1^2 - 2b_0b_1 \cos_\pi \left(\sqrt{\frac{\beta}{4\alpha}} L \right) \right) & \text{for } \alpha, \beta > 0, \\ \frac{4}{\beta} (b_0^2 + b_1^2) & \text{for } \alpha = L = 0, \text{ and } \beta > 0, \\ \frac{L^2}{\alpha} b_0^2 & \text{for } \beta = 0, \alpha > 0, \text{ and } b_1 = b_0, \\ 0 & \text{for } \alpha = \beta = L = 0 \text{ and } b_0 = b_1, \\ \infty & \text{otherwise.} \end{cases} \quad (3.11)$$

We emphasize that $\mathfrak{S}_{0,\beta}$ and $\mathfrak{S}_{\alpha,0}$ can be obtained as Γ -limits of $\mathfrak{S}_{\alpha_n,\beta_n}$ for $\beta_n \searrow 0$ or $\alpha_n \searrow 0$, respectively.

Using $\mathfrak{S}_{\alpha,\beta}$ we can express $\mathcal{J}_{\alpha,\beta}^{\text{1mp}}$ for all $\alpha, \beta \geq 0$, where the cases $\alpha = 0$ or $\beta = 0$ mean that $\dot{\rho} \equiv 0$ or $\dot{a} \equiv 0$, respectively. Moreover $\mathcal{J}_{\alpha,\beta}^{\text{1mp}} = +\infty$, if the set of competitors (a, ρ) providing finite values is empty.

Corollary 3.2 *For all $\alpha, \beta \geq 0$ we have*

$$\mathcal{J}_{\alpha,\beta}^{\text{1mp}}(L^2, a_0, a_1) = \mathfrak{S}_{\alpha,\beta}(L^2, \sqrt{a_0}, \sqrt{a_1}).$$

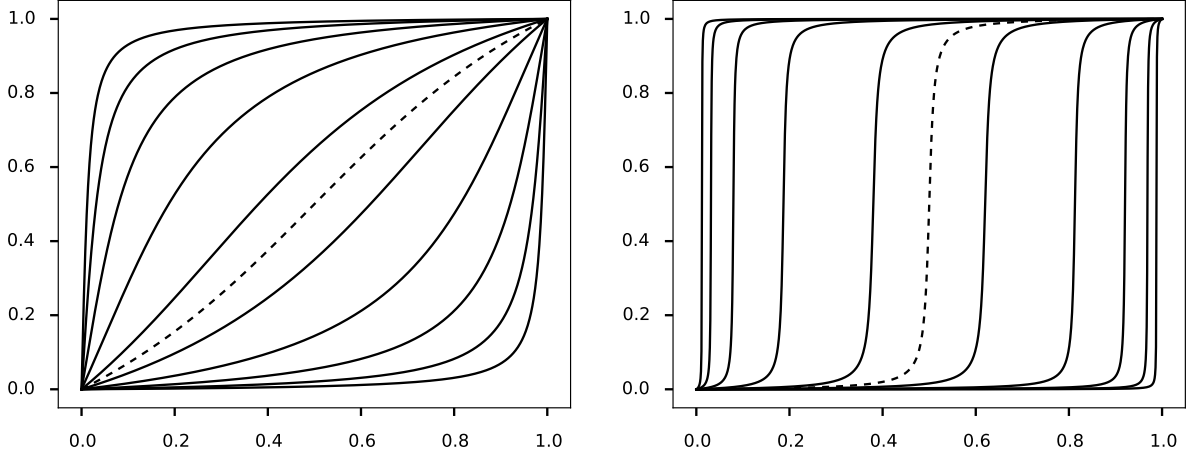


Figure 2: The function $s \mapsto \rho(s)$ in Theorem 3.1 for different ratios a_0/a_1 and $L = \pi/2$ (left) and $L = \pi/1.1$ (right). The dashed curve corresponds to $a_0/a_1 = 1$, while curves above and below satisfy $a_0/a_1 < 1$ and $a_0/a_1 > 1$, respectively.

Proof: We only need to consider the boundary cases.

For $\alpha = \beta = 0$ we have $\dot{a} \equiv 0 \equiv \dot{\rho}$, which implies that $\mathcal{J}_{0,0}^{\text{imp}}$ is finite only for $L = 0$ and $a_0 = a_1$.

For $\alpha = 0$ and $\beta > 0$ we have $\dot{\rho} \equiv 0$ and obtain a finite value only for $L = 0$. Clearly, the infimum of $\int_0^1 \dot{a}^2 / (\beta a) ds$ is given by $4(\sqrt{a(1)} - \sqrt{a(0)})^2 / \beta$, which is the desired result.

The case $\beta = 0$ and $\alpha > 0$ provides $\dot{a} \equiv 0$ and $\dot{\rho} \equiv 1$. Hence, the infimum is $L^2 a_0 / \alpha$ for $a_1 = a_0$ and ∞ otherwise. ■

Example 3.3 (Mass splitting) *At the end of this subsection we give a more complicated example for an optimal curve consisting of two point masses. We want to connect the measures $\mu_0 = a_0 \delta_{x_0}$ and $\mu_1 = a_1 \delta_{x_0} + b_1 \delta_{x_1}$, where $L = |x_0 - x_1| < \pi/2$, i.e. $\cos_\pi(L) = \cos(L) > 0$. So the question is how much of the mass at x_0 is kept there, how much of the mass is used for transport, and how much mass is created at x_1 . We consider the curve*

$$\gamma(s) = a(s)\delta_{x_0} + c(s)\delta_{x(s)} + b(s)\delta_{x_1}$$

with $a(s), b(s), c(s) \geq 0$ and the boundary conditions

$$x(0) = x_0, \quad x(1) = x_1, \quad a(0) + c(0) = a_0, \quad a(1) = a_1, \quad b(0) = 0, \quad c(1) + b(1) = b_1.$$

Choosing $\alpha = 1$ and $\beta = 4$ and optimizing each of the given three curves under their own boundary conditions gives

$$\text{Length}_{1,4}(\gamma)^2 = (\sqrt{a(0)} - \sqrt{a(1)})^2 + c(0) + c(1) - 2\sqrt{c(0)c(1)}\cos_\pi(L) + b(1).$$

From the constraint $c(1) + b(1) = b_1$ and the second last term, we see that it is optimal to choose $c(1)$ as large as possible, namely $c(1) = b_1$ and $b(1) = 0$. In particular, we have no creation at x_1 , i.e. $b \equiv 0$. Setting $c_0 = c(0)$ and eliminating $a(0) = a_0 - c_0$ we find

$$\text{Length}_{1,4}(\gamma)^2 = a_0 - 2\sqrt{a_1}\sqrt{a_0 - c_0} - 2\cos_\pi(L)\sqrt{b_1 c_0} + b_1 + a_1.$$

The minimal value is achieved for the choice $c_0 = a_0 b_1 \cos_\pi(L)^2 / (a_1 + b_1 \cos_\pi(L)^2)$, which means a mass splitting as $0 < c_0 < a_0$. Hence, we have established the estimate

$$D_{1,4}(\mu_0, \mu_1)^2 = D_{1,4}(a_0 \delta_{x_0}, a_1 \delta_{x_0} + b_1 \delta_{x_1})^2 \leq a_0 + a_1 + b_1 - 2\sqrt{a_0(a_1 + b_1 \cos_\pi(|x_0 - x_1|)^2)}.$$

In fact, it will be shown in Example 3.8 that the curve γ is indeed a geodesic curve, i.e. “ \leq ” can be replaced by “ $=$ ”.

3.2 Optimal transport on the cone

The crucial point in the characterization of the distance $D_{\alpha,\beta}$ induced by the Onsager operator $\mathbb{K}_{\alpha,\beta}$ in (3.1), for $\alpha = 1$ and $\beta = 4$, is that the functional $\mathcal{J}_{1,4}^{\text{imp}}$ in (3.6), which gives the cost for optimally transporting a single mass point, is closely related to the metric construction of a *cone over the metric space* $(\Omega, |\cdot|)$. We will briefly explain the construction in this section and refer to [BBI01, Sect. 3.6.2] for more details.

Given the closed and convex domain $\Omega \subset \mathbb{R}^d$ we construct the cone \mathfrak{C}_Ω as the quotient of $\Omega \times [0, \infty[$ over $\Omega \times \{0\}$, i.e.,

$$\mathfrak{C}_\Omega := (\Omega \times [0, \infty[) / (\Omega \times \{0\}).$$

In particular, all points in $\Omega \times \{0\}$ are identified with one point, namely the tip of the cone denoted by \mathfrak{o} . For any $x \in \Omega$ and $r > 0$ the equivalence classes are denoted by $z = [x, r] \in \mathfrak{C}_\Omega$ while for $r = 0$ the equivalence class $[x, 0]$ is equal to \mathfrak{o} .

Motivated by the previous section we define the distance $\mathbf{d}_\mathfrak{C} : \mathfrak{C}_\Omega \times \mathfrak{C}_\Omega \rightarrow [0, \infty[$ on the cone space \mathfrak{C}_Ω as follows:

$$\mathbf{d}_\mathfrak{C}([x_0, r_0], [x_1, r_1])^2 := r_0^2 + r_1^2 - 2r_0 r_1 \cos_\pi(|x_1 - x_0|),$$

where \cos_π is defined as in Theorem 3.1.

For the special case that $\Omega = [0, \ell] \subset \mathbb{R}^2$ with $0 < \ell < 2\pi$, we can visualize $\mathfrak{C}_\Omega = \mathfrak{C}_{[0,\ell]}$ by the two-dimensional sector

$$\Sigma_\ell := \{y = (r \cos x, r \sin x) \in \mathbb{R}^2 \mid r \geq 0, x \in [0, \ell]\},$$

where $y = (0, 0)$ corresponds to the tip \mathfrak{o} . The induced distance is the Euclidian distance restricted to Σ_ℓ , i.e. the geodesic curve between y_0 and y_1 is a straight segment if $|x_1 - x_0| \leq \pi$ while it consists of the two rays connecting $y = 0$ with y_0 and y_1 , respectively, if $\pi \leq |x_1 - x_0| \leq \ell < 2\pi$, see Figure 3. In the case of the traveling mass point discussed in the previous section we identify the Dirac measures $a_i \delta_{x_i}$ with pairs $[x_i, \sqrt{a_i}] \in \mathfrak{C}_\Omega$. Thus, the result of Theorem 3.1 can be reformulated as

$$\mathcal{J}_{1,4}^{\text{imp}}(|x_0 - x_1|^2; a_0, a_1) = \mathbf{d}_\mathfrak{C}([x_0, \sqrt{a_0}], [x_1, \sqrt{a_1}])^2.$$

For general coefficients $\alpha, \beta > 0$ the distance $\mathbf{d}_\mathfrak{C}$ has to be replaced by

$$\mathbf{d}_\mathfrak{C}^{\alpha,\beta}(z_0, z_1) = \mathbf{d}_\mathfrak{C}^{\alpha,\beta}([x_0, r_0], [x_1, r_1]) = \sqrt{\mathfrak{S}_{\alpha,\beta}(|x_1 - x_0|, r_0, r_1)},$$

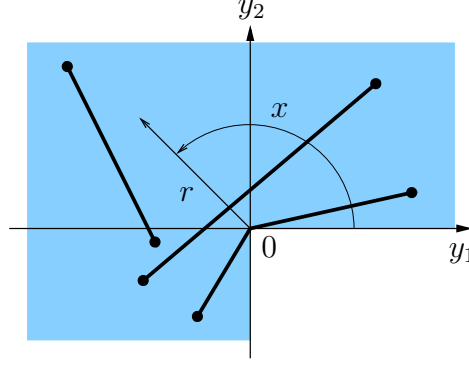


Figure 3: The cone $\mathfrak{C}_{[0,3\pi/2]}$ represented as sector in \mathbb{R}^2 via $(y_1, y_2) = (r \cos x, r \sin x)$ and three geodesic curves. The angle $x = \pi$ is critical for smoothness of geodesic curves.

see (3.11). This distance can be seen as a geodesic distance on the cone \mathfrak{C}_Ω induced by a Riemannian metric (outside of the vertex \mathfrak{o}) given by the tensor

$$\mathbb{G}_{\alpha,\beta}^\mathfrak{C}([x, r]) := \begin{pmatrix} \frac{r^2}{\alpha} I_{\mathbb{R}^d} & 0 \\ 0 & 4/\beta \end{pmatrix} \in \mathbb{R}^{(d+1) \times (d+1)}. \quad (3.12)$$

This fact is already seen in (3.4) if we set $a(s) = r(s)^2$.

3.2.1 Geodesic curves in the cone space

As is shown in [BBI01, Sect. 3.6.2], the pair $(\mathfrak{C}_\Omega, \mathbf{d}_\mathfrak{C})$ is a complete geodesic space, where each two points can be connected by a unique arclength-parameterized geodesic curve. These curves are given by the following geodesic interpolator $Z(s; \cdot, \cdot) : \mathfrak{C}_\Omega \times \mathfrak{C}_\Omega \rightarrow \mathfrak{C}_\Omega$ (recall $z_j = [x_j, r_j]$)

$$Z(s; z_0, z_1) := [X(s; z_0, z_1), R(s; z_0, z_1)] \quad \text{where}$$

$$R(s; z_0, z_1)^2 := (1-s)^2 r_0^2 + s^2 r_1^2 + 2s(1-s)r_0 r_1 \cos_\pi(|x_0 - x_1|),$$

$$X(s; z_0, z_1) := (1-\rho(s; z_0, z_1))x_0 + \rho(s; z_0, z_1)x_1,$$

$$\rho(s; z_0, z_1) := \begin{cases} \frac{1}{|x_1 - x_0|} \arccos\left(\frac{(1-s)r_0 + sr_1 \cos |x_1 - x_0|}{R(s; z_0, z_1)}\right) & \text{for } |x_1 - x_0| < \pi, \\ \frac{1}{2} \left(1 + \text{sign}((1-s)r_0 - sr_1)\right) & \text{for } |x_1 - x_0| \geq \pi. \end{cases} \quad (3.13)$$

Note that in the definition of R there is a “+” in front of the cosine term, while there is a “−” in the distance $\mathbf{d}_\mathfrak{C}$. Moreover, by elementary geometric identities it is easy to see that the formula for ρ is equivalent to the one obtained in Theorem 3.1.

In particular, the curve defined by $\gamma(s) = Z(s; z_0, z_1)$ is a constant speed geodesic curve with respect to the distance $\mathbf{d}_\mathfrak{C}$ connecting z_0 and z_1 , i.e.,

$$\forall 0 \leq s < t \leq 1 : \quad \mathbf{d}_\mathfrak{C}(\gamma(s), \gamma(t)) = |t-s| \mathbf{d}_\mathfrak{C}(z_0, z_1). \quad (3.14)$$

As for the Wasserstein-Kantorovich distance, the geodesic curves in \mathfrak{C}_Ω are key for the construction of the geodesic curves with respect to the Hellinger-Kantorovich distance. We will discuss this in Section 3.4 in detail.

3.2.2 The transport distance modulo reservoirs on the cone space

Using the theory of optimal transport (cf. [AGS05, Vil09]) we can define the Kantorovich–Wasserstein distance $W_{\mathfrak{C}}$ associated with the cone distance $d_{\mathfrak{C}}$ on the set of all nonnegative and finite measures $\mathcal{M}_2(\mathfrak{C}_{\Omega})$ as follows. If $\lambda_0(\mathfrak{C}_{\Omega}) \neq \lambda_1(\mathfrak{C}_{\Omega})$, then we set $W_{\mathfrak{C}}(\lambda_0, \lambda_1) = \infty$ and otherwise we set

$$W_{\mathfrak{C}}(\lambda_0, \lambda_1)^2 := \inf \left\{ \iint_{\mathfrak{C}_{\Omega} \times \mathfrak{C}_{\Omega}} d_{\mathfrak{C}}(z_0, z_1)^2 d\gamma(z_0, z_1) \mid \gamma \in \mathcal{M}(\mathfrak{C}_{\Omega} \times \mathfrak{C}_{\Omega}), \Pi_{\#}^i \gamma = \lambda_i \right\}.$$

Moreover, there the geodesic interpolation (which is not unique in general) can be described using an optimal transport plan γ that is a minimizer in the definition of $W_{\mathfrak{C}}$. We denote the geodesic interpolator by

$$\mathcal{Z}_{\gamma}(s; \lambda_0, \lambda_1) := Z(s; \cdot, \cdot)_{\#} \gamma. \quad (3.15)$$

For the proper handling of creation and annihilation of mass we introduce a modified distance. The modification occurs via a reservoir of mass in the vertex \mathfrak{o} of the cone such that mass is generated from the reservoir and absorbed into it. The following result shows that if we assume that the reservoir is sufficiently big (and in our model for the Hellinger–Kantorovich operator it is in fact infinite), we never have any true transport over distances larger than $\pi/2$ (respectively, $\sqrt{\alpha/\beta}\pi$ in the scaled case), which is only half the critical distance of possible transport.

Proposition 3.4 (Optimal transport in the presence of large reservoirs) *We consider arbitrary measures $\lambda_0, \lambda_1 \in \mathcal{M}_2(\mathfrak{C}_{\Omega})$ with equal masses $\lambda_0(\mathfrak{C}_{\Omega}) = \lambda_1(\mathfrak{C}_{\Omega})$.*

(a) *The function $[0, \infty[\ni \kappa \mapsto w(\lambda_0, \lambda_1, \kappa) := W_{\mathfrak{C}}(\kappa\delta_{\mathfrak{o}} + \lambda_0, \kappa\delta_{\mathfrak{o}} + \lambda_1)$ is nonincreasing.*

(b) *Define the real numbers*

$$\theta_j := \lambda_j(\mathfrak{C}_{\Omega} \setminus \mathfrak{o}), \quad \rho_j := \lambda_j(\{\mathfrak{o}\}), \quad \text{and } \kappa_* = \max\{0, \theta_1 - \rho_0, \theta_0 - \rho_1\},$$

then for all $\kappa \geq \kappa_$ we have $w(\lambda_0, \lambda_1, \kappa) = w(\lambda_0, \lambda_1, \kappa_*)$ with similar transport plans differing only by $(\kappa - \kappa_*)\delta_{\mathfrak{o}, \mathfrak{o}}$.*

(c) *For any optimal transport plan γ connecting $\kappa_*\delta_{\mathfrak{o}} + \lambda_0$ and $\kappa_*\delta_{\mathfrak{o}} + \lambda_1$ we have $\gamma(\mathfrak{N}) = 0$, where*

$$\mathfrak{N} := \{(z_0, z_1) \in \mathfrak{C}_{\Omega} \times \mathfrak{C}_{\Omega} \mid r_0, r_1 > 0, |x_0 - x_1| > \pi/2\},$$

i.e., there is no transport in Ω over distances longer than $\pi/2$.

Proof: *ad (a)* Let $0 \leq \kappa_1 < \kappa_2$ be given and let $\gamma_{\kappa_1}, \gamma_{\kappa_2} \in \mathcal{M}_2(\mathfrak{C}_{\Omega} \times \mathfrak{C}_{\Omega})$ be optimal transport plans for the pairs $\kappa_1\delta_{\mathfrak{o}} + \lambda_i$ and $\kappa_2\delta_{\mathfrak{o}} + \lambda_i$, $i = 0, 1$, respectively. Since $\kappa_2 > \kappa_1$ we can define the transport plan $\widehat{\gamma}_{\kappa_2} = (\kappa_2 - \kappa_1)\delta_{(\mathfrak{o}, \mathfrak{o})} + \gamma_{\kappa_1}$, which satisfies $\Pi_{\#}^i \widehat{\gamma}_{\kappa_2} = \kappa_2\delta_{\mathfrak{o}} + \lambda_i$. Thus, $\widehat{\gamma}_{\kappa_2}$ is an admissible plan for the minimization problem in the definition of $W_{\mathfrak{C}}$ and we obtain the estimate

$$\begin{aligned} w(\lambda_0, \lambda_1, \kappa_2)^2 &= W_{\mathfrak{C}}(\kappa_2\delta_{\mathfrak{o}} + \lambda_0, \kappa_2\delta_{\mathfrak{o}} + \lambda_1)^2 \leq \iint_{\mathfrak{C}_{\Omega} \times \mathfrak{C}_{\Omega}} d_{\mathfrak{C}}(z_0, z_1)^2 d\widehat{\gamma}_{\kappa_2}(z_0, z_1) \\ &= \iint_{\mathfrak{C}_{\Omega} \times \mathfrak{C}_{\Omega}} d_{\mathfrak{C}}(z_0, z_1)^2 d\gamma_{\kappa_1}(z_0, z_1) = w(\lambda_0, \lambda_1, \kappa_1)^2. \end{aligned}$$

Hence, $\kappa \mapsto w(\lambda_0, \lambda_1, \kappa)$ is not increasing.

ad (b) Let γ_κ and γ_{κ_*} denote the optimal transport plans with respect to κ and κ_* , respectively. Similar to (a) we define the measure $\hat{\gamma}_{\kappa_*} = \gamma_\kappa - (\kappa - \kappa_*)\delta_{(\mathfrak{o}, \mathfrak{o})}$. Obviously, $\hat{\gamma}_{\kappa_*}$ satisfies $\Pi_{\#}^i \hat{\gamma}_{\kappa_*} = \lambda_i + \kappa_* \delta_{\mathfrak{o}}$. It remains to show that $\hat{\gamma}_{\kappa_*}$ is nonnegative and hence is an admissible transport plan. Indeed, due to the estimate $\gamma_\kappa(\{\mathfrak{o}\} \times (\mathfrak{C}_\Omega \setminus \{\mathfrak{o}\})) \leq \theta_1$ and the definition of κ_* we obtain

$$\begin{aligned} \gamma_\kappa(\{\mathfrak{o}\} \times \{\mathfrak{o}\}) &= \gamma_\kappa(\{\mathfrak{o}\} \times \mathfrak{C}_\Omega) - \gamma_\kappa(\{\mathfrak{o}\} \times (\mathfrak{C}_\Omega \setminus \{\mathfrak{o}\})) \\ &\geq \kappa + \rho_0 - \theta_1 \geq \kappa - \kappa_*. \end{aligned}$$

Hence, $\hat{\gamma}_{\kappa_*} \geq 0$ and, arguing as for (a) we get $w(\lambda_0, \lambda_1, \kappa_*) \leq w(\lambda_0, \lambda_1, \kappa)$. However, due to the first part of the theorem even equality must hold.

ad (c) As before let γ denote an optimal plan for lifts λ_0 and λ_1 of μ_0 and μ_1 . Assume that $\gamma(\mathfrak{N}) > 0$ such that $\mathbf{d}_{\mathfrak{C}}(z_0, z_1)^2 > r_0^2 + r_1^2$ for γ -a.a. $(z_0, z_1) \in \mathfrak{N}$ with $z_i = [x_i, r_i]$. We aim to construct a new transport plan $\hat{\gamma}$ based on γ giving a strictly lower cost and hence showing the non-optimality of γ .

To this end, we introduce the characteristic function χ of the subset $\mathfrak{N}^c := (\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega) \setminus \mathfrak{N}$. Moreover, we denote by $\tilde{\lambda}_i \in \mathcal{M}_2(\mathfrak{C}_\Omega)$ the marginals of $\tilde{\gamma}_\chi = \chi\gamma$, which are obviously absolutely continuous with respect to λ_i . We denote the densities with ρ_i such that $\tilde{\lambda}_i = \rho_i \lambda_i$. In particular, for λ_i -a.e. $z \in \mathfrak{C}_\Omega$ we have that $0 \leq \rho_i \leq 1$.

We define the measure $\hat{\gamma}$

$$\hat{\gamma}(\mathrm{d}z_0, \mathrm{d}z_1) = \tilde{\gamma}_\chi(\mathrm{d}z_0, \mathrm{d}z_1) + (1 - \rho_0)\lambda_0(\mathrm{d}z_0)\delta_{\mathfrak{o}}(\mathrm{d}z_1) + \delta_{\mathfrak{o}}(\mathrm{d}z_0)(1 - \rho_1)\lambda_1(\mathrm{d}z_1).$$

We easily check that the marginals of $\hat{\gamma}$ are given by $\hat{\lambda}_i = \lambda_i + \kappa \delta_{\mathfrak{o}}$, $i = 0, 1$, where $\kappa > 0$ is given by $\kappa = (\gamma - \tilde{\gamma}_\chi)(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega)$. In particular, $\hat{\lambda}_i$ is an admissible lift for μ_i .

It remains to show that $\hat{\gamma}$ has a strictly lower cost than γ . We compute

$$\begin{aligned} \iint_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} \mathbf{d}_{\mathfrak{C}}(z_0, z_1)^2 \mathrm{d}\hat{\gamma} &= \iint_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} \mathbf{d}_{\mathfrak{C}}(z_0, z_1)^2 \mathrm{d}\tilde{\gamma}_\chi + \int_{\mathfrak{C}_\Omega} r_0^2(1 - \rho_0) \mathrm{d}\lambda_0 + \int_{\mathfrak{C}_\Omega} r_1^2(1 - \rho_1) \mathrm{d}\lambda_1 \\ &= \iint_{\mathfrak{N}^c} \mathbf{d}_{\mathfrak{C}}(z_0, z_1)^2 \mathrm{d}\gamma + \iint_{\mathfrak{N}} (r_0^2 + r_1^2) \mathrm{d}\gamma \\ &< \iint_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} \mathbf{d}_{\mathfrak{C}}(z_0, z_1)^2 \mathrm{d}\gamma. \end{aligned}$$

Thus, γ cannot be optimal and any optimal transport plan has to vanish on \mathfrak{N} . \blacksquare

Using the above proposition we may define a new distance W_{rsv} on $\mathcal{M}_2(\mathfrak{C}_\Omega)$ that assumes that the reservoir is always big enough. Indeed, defining

$$W_{\text{rsv}}(\lambda_0, \lambda_1) := \inf_{\kappa > 0} W_{\mathfrak{C}}(\lambda_0 + \kappa \delta_{\mathfrak{o}}, \lambda_1 + \kappa \delta_{\mathfrak{o}}) = W_{\mathfrak{C}}(\lambda_0 + \kappa_* \delta_{\mathfrak{o}}, \lambda_1 + \kappa_* \delta_{\mathfrak{o}}),$$

where κ_* is given as in Proposition 3.4(b).

3.3 The Hellinger–Kantorovich distance

We can now easily define a distance for measures on Ω by lifting measures $\mu_j \in \mathcal{M}(\Omega)$ to measures on $\mathcal{M}_2(\mathfrak{C}_\Omega)$ and projecting back measures from $\mathcal{M}_2(\mathfrak{C}_\Omega)$ into $\mathcal{M}(\Omega)$. We define

the projection $\mathfrak{P} : \mathcal{M}_2(\mathfrak{C}_\Omega) \rightarrow \mathcal{M}(\Omega)$ via

$$\int_{\Omega} \phi(x) d\mathfrak{P}\lambda = \int_{\mathfrak{C}_\Omega} r^2 \phi(x) d\lambda([x, r]) \quad \text{for all } \forall \phi \in C^0(\Omega).$$

In the last formula we use that for $r > 0$ the equivalence class $[x, r]$ uniquely determines x and r and that the prefactor r^2 makes the function $\Phi : [x, r] \mapsto r^2 \phi(x)$ continuous if we set $\Phi(\mathfrak{o}) = 0$. In the case $\mu = \mathfrak{P}\lambda$ we call λ a lift of μ .

The first and most intuitive result on the distance $D_{1,4}$ induced by the Onsager operator $\mathbb{K}_{1,4}$ in (3.1) is the following formula, which we formulate as a definition first and then show that it equals the distance $D_{1,4}$.

Definition 3.5 *The Hellinger–Kantorovich distance on $\mathcal{M}(\Omega)$ is defined as*

$$\mathbf{HK}(\mu_0, \mu_1) = \min \left\{ W_{\mathfrak{C}}(\lambda_0, \lambda_1) \mid \mathfrak{P}\lambda_0 = \mu_0, \mathfrak{P}\lambda_1 = \mu_1 \right\}. \quad (3.16)$$

Before proving the identity $\mathbf{HK} = \mathbf{HK}_{1,4} = D_{1,4}$ in Section 4, we collect some properties of \mathbf{HK} . First, we emphasize that the projection \mathfrak{P} does not see the reservoirs at \mathfrak{o} , hence the above formula already includes arbitrary reservoirs according to Proposition 3.4.

Next, let us remark that \mathbf{HK} satisfies an important scaling invariance: Let $\vartheta : \mathfrak{C}_\Omega \times \mathfrak{C}_\Omega \rightarrow]0, \infty[$ be a Borel map and define the dilation function $h_\vartheta : \mathfrak{C}_\Omega \times \mathfrak{C}_\Omega \rightarrow \mathfrak{C}_\Omega \times \mathfrak{C}_\Omega$ via

$$h_\vartheta(z_0, z_1) = \left(\left[x_0, \frac{r_0}{\vartheta(z_0, z_1)} \right], \left[x_1, \frac{r_1}{\vartheta(z_0, z_1)} \right] \right) \quad \text{for } z_i = [x_i, r_i]. \quad (3.17)$$

Then, given any transport plan $\gamma \in \mathcal{M}_2(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega)$ we define the dilated plan $\gamma_\vartheta = (h_\vartheta)_\#(\vartheta^2 \gamma)$ in $\mathcal{M}(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega)$. Letting λ_i and λ_i^ϑ denote the marginals of γ and γ_ϑ we have that

$$\iint_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} d_{\mathfrak{C}}(z_0, z_1)^2 d\gamma = \iint_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} d_{\mathfrak{C}}(z_0, z_1)^2 d\gamma_\vartheta \quad \text{and} \quad \mathfrak{P}\lambda_i = \mathfrak{P}\lambda_i^\vartheta. \quad (3.18)$$

In particular, we can always assume that the transport plans γ and the lifts λ_i are probability measures, e.g. by setting $\vartheta \equiv (\gamma(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega))^{-1/2}$.

The main result of this section is the following structural theorem. For a full proof we refer to [LMS15], where a more general case is considered. In particular, there Ω is replaced by general complete geodesic spaces. However, because of the strong relevance of part (v) for the subsequent applications, we present a full proof of the identity $\mathbf{HK} = D_{1,4}$ in Section 4.

Theorem 3.6 (Properties of \mathbf{HK}) *The distance $\mathbf{HK} : \mathcal{M}(\Omega) \times \mathcal{M}(\Omega) \rightarrow [0, \infty[$ has the following properties:*

- i) *For each pair μ_0, μ_1 there exists an optimal pair λ_0, λ_1 of lifts;*
- ii) *For all measures μ_0, μ_1 the upper bound $\mathbf{HK}(\mu_0, \mu_1)^2 \leq \mu_0(\Omega) + \mu_1(\Omega)$ is satisfied;*
- iii) *$(\mathcal{M}(\Omega), \mathbf{HK})$ is a complete and separable metric space;*
- iv) *The topology induced by \mathbf{HK} coincides with the weak topology on $\mathcal{M}(\Omega)$;*
- v) *The distance \mathbf{HK} is induced by the Onsager operator $\mathbb{K}_{1,4}$.*

3.3.1 Consistency of above formulas with distance of Dirac masses

We come back to the example of the optimal transport and absorption/desorption of two point masses in Subsection 3.1 and discuss the consistency of the above formulas. Let $\mu_0 = a_0\delta_{x_0}$ and $\mu_1 = a_1\delta_{x_1}$ denote two Dirac masses such that $a_0, a_1 > 0$. We consider lifts $\lambda_0, \lambda_1 \in \mathcal{M}_2(\mathfrak{C}_\Omega)$ of the particular form

$$\lambda_0 = \left(\kappa + \frac{a_1}{r_1^2}\right)\delta_{\mathfrak{o}} + \frac{a_0}{r_0^2}\delta_{[x_0, r_0]}, \quad \text{and} \quad \lambda_1 = \left(\kappa + \frac{a_0}{r_0^2}\right)\delta_{\mathfrak{o}} + \frac{a_1}{r_1^2}\delta_{[x_1, r_1]},$$

where $\kappa \geq 0$ and $r_i > 0$ are arbitrary but fixed constants. In particular, we have equal mass $\lambda_0(\mathfrak{C}_\Omega) = \lambda_1(\mathfrak{C}_\Omega)$ and $\mathfrak{P}\lambda_i = \mu_i$, i.e., λ_i is indeed a lift for μ_i .

The possible transport plans $\gamma \in \mathcal{M}_2(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega)$ are uniquely characterized by the value $g := \gamma(\{[x_0, r_0], [x_1, r_1]\}) \in [0, \min\{a_0/r_0^2, a_1/r_1^2\}]$, where the interval boundaries correspond to complete absorption/desorption and complete transport.

Denoting $z_i = [x_i, r_i]$ we find

$$\begin{aligned} \iint_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} \mathbf{d}_{\mathfrak{C}}(z_0, z_1)^2 d\gamma(z_0, z_1) &= \frac{a_0}{r_0^2} \mathbf{d}_{\mathfrak{C}}(z_0, \mathfrak{o})^2 + \frac{a_1}{r_1^2} \mathbf{d}_{\mathfrak{C}}(z_1, \mathfrak{o})^2 \\ &\quad + g[\mathbf{d}_{\mathfrak{C}}(z_0, z_1)^2 - \mathbf{d}_{\mathfrak{C}}(z_0, \mathfrak{o})^2 - \mathbf{d}_{\mathfrak{C}}(z_1, \mathfrak{o})^2] \\ &= a_0 + a_1 - 2gr_0r_1\cos\pi(|x_0 - x_1|). \end{aligned}$$

To get the optimal cost we have to minimize with respect to $g \in [0, \min\{a_0/r_0^2, a_1/r_1^2\}]$: For $L := |x_0 - x_1| > \pi/2$ the optimal value is $g = 0$, which corresponds to the pure Hellinger reaction case. For $L = \pi/2$ any g is possible giving a convex set of optimal plans. In fact, it will be shown in Section 5.2 that any pair of lifts is optimal in this case. Hence, the case $L \geq \pi/2$ yields $\mathbf{W}_{\mathfrak{C}}(\lambda_0, \lambda_1) = \sqrt{a_0 + a_1} = \mathbf{HK}(a_0\delta_{x_0}, a_1\delta_{x_1})$ as desired. For $L < \pi/2$ we have to choose $g = \min\{a_0/r_0^2, a_1/r_1^2\}$, i.e., the maximal value. With this we obtain

$$\mathbf{W}_{\mathfrak{C}}(\lambda_0, \lambda_1)^2 = a_0 + a_1 - 2g_*(r_0, r_1)\cos(L),$$

where $g_*(r_0, r_1) := \min\{a_0\frac{r_1}{r_0}, a_1\frac{r_0}{r_1}\}$. In particular, different lifts $\lambda_i = \lambda_i(r_0, r_1)$ give different costs. However, an easy calculation shows that for $r_1/r_0 = \sqrt{a_1/a_0}$ an optimal value is achieved, such that $\mathbf{W}_{\mathfrak{C}}(\lambda_0, \lambda_1)^2 = a_0 + a_1 - 2\sqrt{a_0a_1}\cos(L) = \mathbf{HK}(a_0\delta_{x_0}, a_1\delta_{x_1})^2$.

For calculating the distance \mathbf{HK} the particular choice of an optimal lift is not important, but we will see in Section 5.2 that in the case $L = \pi/2$ different lifts may give rise to different geodesic curves. Hence, we highlight here that even in the trivial case $L < \pi/2$ there are many optimal lifts. E.g. for $a_1 = a_0 > 0$ any $\eta \in \mathcal{M}_2([0, \infty[)$ with $\int_0^\infty r^2 d\eta = a_0$ defines optimal lifts $\lambda_j = \delta_{x_j} \otimes \eta$.

3.3.2 Logarithmic-entropy transport functional

In this subsection we give the formula for the distance via a minimization problem and discuss a few of its properties, in particular its consistency with the distance of Dirac masses. We do this for the case of general positive α and β .

Using the Boltzmann function $F_B(\rho) = \rho \log \rho - \rho + 1 \geq 0$ with $F'_B(\rho) = \log \rho$ and $F_B(\rho) = 0$ for $\rho = 0$ we define the Hellinger–Kantorovich functional for any $\mu_0, \mu_1 \in \mathcal{M}(\Omega)$

as follows. For $\eta \in \mathcal{M}(\Omega \times \Omega)$ we define the marginals $\eta_j = \Pi_{\#}^j \eta$ and assume $\eta_0 \ll \mu_0$ and $\eta_1 \ll \mu_1$ and define the Hellinger–Kantorovich entropy-transport functional via

$$\mathcal{E}\mathcal{T}_{\alpha,\beta}(\eta; \mu_0, \mu_1) := \frac{4}{\beta} \int_{\Omega} F_B\left(\frac{d\eta_0}{d\mu_0}\right) d\mu_0 + \frac{4}{\beta} \int_{\Omega} F_B\left(\frac{d\eta_1}{d\mu_1}\right) d\mu_1 + \int_{\Omega \times \Omega} c_{\alpha,\beta}(|x_0 - x_1|) d\eta,$$

where the cost function $c_{\alpha,\beta}$ is given by

$$c_{\alpha,\beta}(L) := \begin{cases} -\frac{8}{\beta} \log \left(\cos \left(\sqrt{\beta/(4\alpha)} L \right) \right) & \text{for } L < \pi \sqrt{\alpha/\beta}, \\ \infty & \text{for } L \geq \pi \sqrt{\alpha/\beta}. \end{cases}$$

We see that $\mathcal{E}\mathcal{T}_{1,4}(\cdot; \mu_0, \mu_1)$ is a convex functional, thus it is easy to find minimizers. The following characterization is proved in full detail in [LMS15]. Here we will only motivate the construction by giving some examples.

Theorem 3.7 (Characterization of $\mathbf{HK}_{\alpha,\beta}$ via minimization) *For $\alpha, \beta > 0$ the distance induced by the Onsager operator $\mathbb{K}_{\alpha,\beta}$ is given as follows:*

$$\mathbf{HK}_{\alpha,\beta}(\mu_0, \mu_1)^2 = \mathbf{ET}_{\alpha,\beta}(\mu_0, \mu_1) := \inf \left\{ \mathcal{E}\mathcal{T}_{\alpha,\beta}(\eta; \mu_0, \mu_1) \mid \eta \in \mathcal{M}(\Omega \times \Omega), \eta_j \ll \mu_j \right\}.$$

For every pair (μ_0, μ_1) at least one minimizer η exists, which we call a calibration measure for this pair.

Moreover, an optimal calibration measure η satisfies for $\varrho_i := d\eta_i/d\mu_i$ the following optimality conditions

$$\begin{aligned} |x_0 - x_1| &< \pi \sqrt{\alpha/\beta} && \text{for } \eta\text{-a.e. } (x_0, x_1) \in \Omega \times \Omega, \\ \varrho_0(x_0)\varrho_1(x_1) &\leq \cos \left(\sqrt{\beta/(4\alpha)} |x_0 - x_1| \right)^2 && \text{for } \mu_0\text{-a.e. } x_0 \in \Omega \text{ and } \mu_1\text{-a.e. } x_1 \in \Omega, \\ \varrho_0(x_0)\varrho_1(x_1) &= \cos \left(\sqrt{\beta/(4\alpha)} |x_0 - x_1| \right)^2 > 0 && \text{for } \eta\text{-a.e. } (x_0, x_1) \in \Omega \times \Omega. \end{aligned}$$

In the framework of this paper, the relevance of this new characterization of $\mathbf{D}_{1,4} = \mathbf{HK}$ is that the minimization of $\mathcal{E}\mathcal{T}_{1,4}$ is much simpler than the characterization of \mathbf{HK} in terms of lifts to the cone space. Finding the optimal lifts and the calculating the optimal transport on the cone space is certainly more involved. In [LMS15], it is shown that $\mathcal{E}\mathcal{T}_{1,4}$ has a much stronger intrinsic value and it proves an essential tool for establishing the results in Theorem 3.6.

Example 3.8 (Mass splitting, part 2) *We return to Example 3.3 where we calculated the distance between*

$$\mu_0 = a_0 \delta_{x_0} \quad \text{and} \quad \mu_1 = a_1 \delta_{x_0} + b_1 \delta_{x_1}$$

with $L = |x_0 - x_1| < \pi$ and $\alpha = 1, \beta = 4$. We show that the formulation in Theorem 3.7 indeed gives the same cost. Since the marginals η_j have to have a density with respect to μ_j and since η_0 and η_1 must have equal mass, we consider

$$\eta_0 = e_0 \delta_{x_0} \quad \text{and} \quad \eta_1 = (e_0 - e_1) \delta_{x_0} + e_1 \delta_{x_1}$$

with $e_0, e_1 \geq 0$. Using the formula in Theorem 3.7 yields for $\mathbf{ET} = \mathbf{ET}_{1,4}$ and $\mathbf{c} = \mathbf{c}_{1,4}$

$$\mathbf{ET}(\mu_0, \mu_1) = \inf \left\{ F_B\left(\frac{e_0}{a_0}\right)a_0 + F_B\left(\frac{e_0 - e_1}{a_1}\right)a_1 + F_B\left(\frac{e_1}{b_1}\right)b_1 + e_1 \mathbf{c}(L) \mid e_0 \geq e_1 \geq 0 \right\}.$$

This infimum can be evaluated explicitly and we obtain

$$\mathbf{ET}(\mu_0, \mu_1) = \mathbf{HK}(\mu_0, \mu_1)^2 = a_0 + a_1 + b_1 - 2\sqrt{a_0(a_1 + b_1 \cos(|x_0 - x_1|)^2)},$$

which is the same as in Example 3.3.

3.3.3 Reduction to special lifts

The characterization of the Hellinger–Kantorovich distance in terms of the logarithmic-entropy transport functional gives rise to another helpful property: To calculate $\mathbf{HK}(\mu_0, \mu_1)$ it is sufficient to consider lifts λ_i of a special form only. Indeed, assume that $\eta \in \mathcal{M}(\Omega \times \Omega)$ is a minimizer of $\mathcal{E}\mathcal{T}_{1,4}$ for given μ_0 and μ_1 and consider for $\eta_i = \Pi_{\#}^i \eta$ the Lebesgue decomposition $\mu_i = \sigma_i \eta_i + \mu_i^{\perp}$. Then, the transport plan $\gamma_{\eta} \in \mathcal{M}(\mathfrak{C}_{\Omega} \times \mathfrak{C}_{\Omega})$ defined by

$$\begin{aligned} \gamma_{\eta}(\mathrm{d}z_0, \mathrm{d}z_1) &= \delta_{\sqrt{\sigma_0(x_0)}}(\mathrm{d}r_0) \delta_{\sqrt{\sigma_1(x_1)}}(\mathrm{d}r_1) \eta(\mathrm{d}x_0, \mathrm{d}x_1) \\ &\quad + \delta_1(\mathrm{d}r_0) \mu_0^{\perp}(\mathrm{d}x_0) \delta_{\circ}(\mathrm{d}z_1) + \delta_{\circ}(\mathrm{d}z_0) \delta_1(\mathrm{d}r_1) \mu_1^{\perp}(\mathrm{d}x_1) \end{aligned}$$

and the associated lifts $\lambda_i = \Pi_{\#}^i \gamma_{\eta}$ are optimal in the Definition 3.5 for \mathbf{HK} , see [LMS15, Thm. 7.21] for the proof. In particular, we can restrict the analysis to lifts of μ_j characterized by a single positive function $\widehat{r}_j > 0$ on Ω , namely

$$\begin{aligned} \mathfrak{L}(\mu, \widehat{r}, \kappa) &= \kappa \delta_{\circ} + \frac{1}{\widehat{r}(x)^2} \delta_{\widehat{r}(x)}(\mathrm{d}r) \mu(\mathrm{d}x), \quad \text{such that} \\ \int_{\mathfrak{C}_{\Omega}} \Phi(z) \mathrm{d}\mathfrak{L}(\mu, \widehat{r}, \kappa) &= \kappa \Phi(\circ) + \int_{\Omega} \frac{\Phi([x, \widehat{r}(x)])}{\widehat{r}(x)^2} \mathrm{d}\mu \quad \text{for all } \Phi \in C^0(\mathfrak{C}_{\Omega}). \end{aligned}$$

We collect this observation in the following result.

Proposition 3.9 (HK via special lifts) *We have the equivalent characterization*

$$\mathbf{HK}(\mu_0, \mu_1) = \min \left\{ W_{\mathfrak{C}}(\mathfrak{L}(\mu_0, \widehat{r}_0, \kappa_0), \mathfrak{L}(\mu_1, \widehat{r}_1, \kappa_1)) \mid \kappa_j \geq 0, \widehat{r}_j > 0 \right\}. \quad (3.19)$$

Moreover, it is sufficient to consider transport plans $\gamma \in \mathcal{M}_2(\mathfrak{C}_{\Omega} \times \mathfrak{C}_{\Omega})$ of the form

$$\begin{aligned} \gamma &= \delta_{\widehat{r}_0(x_0)}(\mathrm{d}r_0) \eta_0(\mathrm{d}x_0) \delta_{\circ}(\mathrm{d}z_1) + \delta_{\circ}(\mathrm{d}z_0) \delta_{\widehat{r}_1(x_1)}(\mathrm{d}r_1) \eta_1(\mathrm{d}x_1) \\ &\quad + \delta_{\widehat{r}_0(x_0)}(\mathrm{d}r_0) \delta_{\widehat{r}_1(x_1)}(\mathrm{d}r_1) \eta(\mathrm{d}x_0, \mathrm{d}x_1) \end{aligned}$$

for positive functions $\widehat{r}_i : \Omega \rightarrow]0, \infty[$ and measures $\eta_i \in \mathcal{M}(\Omega)$ and $\eta \in \mathcal{M}(\Omega \times \Omega)$.

Using the definition of $W_{\mathfrak{C}}$ in terms of $\mathrm{d}_{\mathfrak{C}}$ and the form of the lifts, the functional in (3.19) can be written as

$$\mathcal{D}(\eta, \widehat{r}_0, \widehat{r}_1; \mu_0, \mu_1) := \mu_0(\Omega) + \mu_1(\Omega) - \int_{\Omega \times \Omega} 2\widehat{r}_0(x_0)\widehat{r}_1(x_1) \cos_{\pi/2} |x_0 - x_1| \mathrm{d}\eta(x_0, x_1),$$

and the following characterization of \mathbf{HK} follows:

$$\mathbf{HK}(\mu_0, \mu_1) = \min \left\{ \mathcal{D}(\eta, \widehat{r}_0, \widehat{r}_1; \mu_0, \mu_1) \mid \eta \in \mathcal{M}(\Omega \times \Omega), \Pi_{\#}^j \eta = \widehat{r}_j^2 \mu_j + \mu_j^{\perp} \right\}. \quad (3.20)$$

We emphasize that not all optimal transport plans are of the form depicted in Proposition 3.9. In particular, using again the example of two mass-points we show in Section 5.2 that in the case of the critical distance $|x_0 - x_1|$ lifts are quite arbitrary.

3.3.4 Recovering the Hellinger and Wasserstein-Kantorovich distances

The log-entropy formulation of the Hellinger–Kantorovich distance is well suited to pass to the limits $\alpha \rightarrow 0$ or $\beta \rightarrow 0$.

Since apart from the prefactor $1/\beta$ the functional only depends on β/α , we can set $\alpha = 1$ and consider the case $\beta \rightarrow 0$. For the cost functional we obtain the expansion

$$c_{1,\beta}(x_0, x_1) = |x_1 - x_0|^2 + O(\beta)$$

uniformly on $\Omega \times \Omega$, which is compact. Hence the linear transport functional converges to the Kantorovich functional for the usual Euclidian cost function. Simultaneous the entropic terms blow up, which means that in the limit $\beta = 0$, we obtain the condition $\eta_j = \mu_j$. Thus, we expect to obtain the Wasserstein distance in the limit, i.e. $\mathbf{HK}_{1,0}(\mu_0, \mu_1) = W(\mu_0, \mu_1)$.

Keeping $\beta = 4$ fixed and considering $\alpha \rightarrow 0$ we obtain

$$c_{\alpha,4}(x_0, x_1) \rightarrow \begin{cases} 0 & \text{for } x_0 = x_1, \\ \infty & \text{for } x_0 \neq x_1. \end{cases}$$

Thus, optimal calibration measures for $\alpha = 0$ will have support on the diagonal $\{(x, x) \in \Omega \times \Omega \mid x \in \Omega\}$, such that the transport cost equals 0 and that $\nu := \eta_0 = \eta_1$. Minimizing the sum of the two entropic terms with respect to ν we obtain the unique solution ν from the optimality condition $\frac{d\nu}{d\mu_0} \frac{d\nu}{d\mu_1} \equiv 1$ and we find $\mathbf{HK}_{0,4}(\mu_0, \mu_1) = D_{\text{Hell}}(\mu_0, \mu_1) = \|\sqrt{\mu_1} - \sqrt{\mu_0}\|_{L^2}$.

3.4 Geodesic curves induced by optimal transport plans

Let $\mu_0, \mu_1 \in \mathcal{M}(\Omega)$ be two given measures. The geodesic curves with respect to the Hellinger–Kantorovich distance \mathbf{HK} are induced by the geodesic curves in the underlying cone space.

More precisely, the construction of the geodesic curve $s \mapsto \mu(s)$ is based on the geodesic interpolator Z defined in (3.13): Let $\lambda_0 \in \mathcal{M}_2(\mathfrak{C}_\Omega)$ and $\lambda_1 \in \mathcal{M}_2(\mathfrak{C}_\Omega)$ be optimal lifts for μ_0 and μ_1 , respectively, and let $\gamma \in \mathcal{M}_2(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega)$ be the associated optimal transport plan. Then, a geodesic curve $\mu(s) = \mathcal{G}(s; \mu_0, \mu_1)$ is obtained via the projection of the geodesic curve for λ_0 and λ_1 in $\mathcal{M}_2(\mathfrak{C}_\Omega)$ via

$$\mu(s) = \mathcal{G}(s; \mu_0, \mu_1) := \mathfrak{P}\lambda(s) \quad \text{with} \quad \lambda(s) = Z(s; \cdot, \cdot)_\# \gamma. \quad (3.21)$$

Note that since the optimal transport plan γ is not necessarily unique, the geodesics in $\mathcal{M}(\Omega)$ are also not necessarily unique:

Example 3.10 (i) On $\Omega =]-2, 2[^2$ we consider the measure $\mu_0 = \delta_{(-1,0)} + \delta_{(1,0)}$ and μ_1 is the line measure concentrated in $\{0\} \times]-1, 1[$. Due to the high symmetry of the problem, it is easy to see that there are infinitely many optimal transport plans, which give rise to different geodesic curves.

(ii) Consider case of two mass points $\mu_i = a_i \delta_{x_i}$ with $|x_0 - x_1| = \pi/2$. It is easy to see that in this case $\mu(s) = a(s) \delta_{x(s)}$ with $x(s) = (1 - \rho(s))x_0 + \rho(s)x_1$ and $a(s)$ and $\rho(s)$ as in Theorem 3.1 and $\tilde{\mu}(s) = (1-s)^2 a_0 \delta_{x_0} + s^2 a_1 \delta_{x_1}$ are both geodesic curves. However, the situation is even more complicated since even along a geodesic curve...

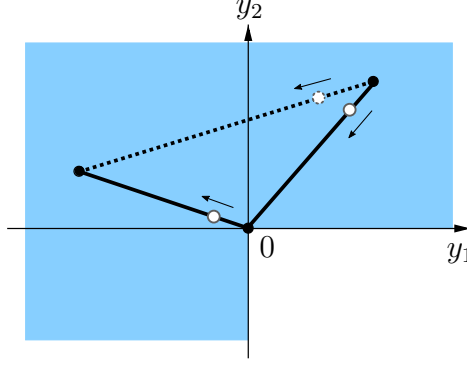


Figure 4: Cone geodesic (dotted) for $z_0 = [x_0, \sqrt{a_0}]$ and $z_1 = [x_1, \sqrt{a_1}]$ compared to Hellinger–Kantorovich geodesic (solid) for $\mu_0 = a_0\delta_{x_0}$ and $\mu_1 = a_1\delta_{x_1}$ in the case $|x_0 - x_1| > \pi/2$. The Hellinger–Kantorovich geodesic consists of two parts: one part is going to the reservoir (absorption), while the other one is simultaneously coming from the reservoir (generation).

Theorem 3.11 *The curve $s \mapsto \mu(s)$ defined in (3.21) is a constant-speed geodesic with respect to the Hellinger–Kantorovich distance \mathbf{HK} , i.e.,*

$$\mathbf{HK}(\mu(s), \mu(t)) = |t-s|\mathbf{HK}(\mu_0, \mu_1) \quad \text{for all } 0 \leq s < t \leq 1.$$

Proof: Fix $0 \leq s < t \leq 1$ and let $\gamma \in \mathcal{M}_2(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega)$ denote the optimal transport plan. We define the map $\Pi_{st} : \mathfrak{C}_\Omega^2 \rightarrow \mathfrak{C}_\Omega^2$ via $\Pi_{st}(z_0, z_1) = (Z(s; z_0, z_1), Z(t; z_0, z_1))$ and introduce the transport plan $\gamma_{st} = (\Pi_{st})_\# \gamma$ whose marginals are given by $\lambda(s)$ and $\lambda(t)$, respectively. In particular, we have the upper estimate

$$\mathbf{HK}(\mu(s), \mu(t)) \leq W_{\mathfrak{C}}(\lambda(s), \lambda(t)) \leq \left(\iint_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} d_{\mathfrak{C}}(z_0, z_1)^2 d\gamma_{st} \right)^{1/2}.$$

However, using the definition of the γ_{st} and that Z is the geodesic interpolator in \mathfrak{C}_Ω we obtain

$$\mathbf{HK}(\mu(s), \mu(t)) \leq |s-t|\mathbf{HK}(\mu_0, \mu_1) \quad \text{for all } 0 \leq s < t \leq 1. \quad (3.22)$$

To see that actually equality holds we use the triangle inequality and (3.22) to find

$$\begin{aligned} \mathbf{HK}(\mu_0, \mu_1) &\leq \mathbf{HK}(\mu_0, \mu_s) + \mathbf{HK}(\mu_s, \mu_t) + \mathbf{HK}(\mu_t, \mu_1) \\ &\leq (s + (t-s) + (1-t))\mathbf{HK}(\mu_0, \mu_1) = \mathbf{HK}(\mu_0, \mu_1). \end{aligned}$$

Thus, all inequalities are equalities, which proves theorem. \blacksquare

4 Equivalence to the dynamical formulation

In this subsection we provide the proof of part (v) of Theorem 3.6 and show the equivalence of the two definitions of the Hellinger–Kantorovich distance, namely the formulation via lifts and optimal transport on the cone space and the dynamical formulation given by the Onsager operator $\mathbb{K} := \mathbb{K}_{1,4}$, i.e. $\mathbf{D}_{1,4} = \mathbf{HK}$ with $\mathbf{D}_{1,4}$ from (3.1) and \mathbf{HK} from (3.16).

The proof is based on the characterization of absolutely continuous curves and their metric derivative with respect to \mathbf{HK} . In particular, we show in Theorem 4.5 that each

absolutely continuous curve whose metric derivative is square integrable satisfies the modified continuity equation in the definition of $D_{1,4}$ in the distributional sense for a suitable vector and scalar field Ξ and ξ . Moreover, the $L^2(d\mu)$ -norms of Ξ and ξ provide a lower bound for the metric derivative.

Vice versa we prove in Theorem 4.6 that a continuous solution $t \mapsto \mu(t)$ of the modified continuity equation for given vector and scalar fields Ξ and ξ is absolutely continuous with respect to \mathbf{HK} and the $L^2(d\mu)$ -norms give an upper estimate for the metric derivative.

Finally, Theorem 3.6(v) is proven at the end of this subsection.

We recall that a curve $[0, 1] \ni t \mapsto u(t)$ in a metric space (\mathbf{Y}, D) is called absolutely continuous if there exists a function $m \in L^1(0, 1)$ such that

$$D(u(s), u(t)) \leq \int_s^t m(r) dr \quad \text{for all } 0 \leq s < t \leq 1. \quad (4.1)$$

We write $u \in AC^p(0, 1; (\mathbf{Y}, D))$ if $m \in L^p(0, 1)$ for $p \in [1, \infty]$. Moreover, among all possible choices for m there exists a minimal one, which is given by the metric derivative, see e.g. [AGS05, Sect. 1.1]

$$|\dot{u}|_D(t) := \lim_{s \rightarrow t} \frac{D(u(t), u(s))}{|t - s|}. \quad (4.2)$$

In particular, for any $u \in AC^p(0, 1; (\mathbf{Y}, D))$ the metric derivative exists for a.a. $t \in]0, 1[$ and satisfies $|\dot{u}|_D \in L^p(0, 1)$ as well as $|\dot{u}|_D \leq m$ a.e. in $[0, 1]$ for all m in (4.1).

We start with a result for the regular case, i.e. the vector and scalar fields Ξ and ξ are sufficiently smooth. The proof of the following result can be found in [Man07] where representation formulas for solutions of the inhomogeneous continuity equation

$$\frac{d}{dt}\mu + \operatorname{div}(\Xi\mu) = 4\xi\mu \quad (4.3)$$

based on dynamic plans are proved. We will briefly recall these results, however, since the cone structure did not play a role in [Man07] we will reinterpret the results in our setting. In the following we understand weak convergence in the space of measures as convergence against bounded and continuous functions. Moreover, a curve $s \mapsto \mu(s) \in \mathcal{M}(\Omega)$ is called weakly continuous if and only if $\mu(s)$ weakly converges to $\mu(t)$ in $\mathcal{M}(\Omega)$ for $s \rightarrow t$.

Proposition 4.1 ([Man07], **Prop. 3.6**) *Assume that $\Xi \in L^1(0, T; W^{1,\infty}(\Omega; \mathbb{R}^d))$ and $\xi \in C([0, T] \times \Omega)$ is locally Lipschitz with respect to the spatial variable. Then, for any $\mu_0 \in \mathcal{M}(\Omega)$ there exists a unique, weakly continuous solution $t \mapsto \mu(t)$ of (4.3) with $\mu(0) = \mu_0$.*

Moreover, for an arbitrary lift $\lambda_0 \in \mathcal{M}(\mathfrak{C}_\Omega)$ of μ_0 the curve defined by

$$\lambda(t) = [X(t; \cdot), R(t; \cdot)]_\# \lambda_0 \in \mathcal{M}(\mathfrak{C}_\Omega), \quad (4.4)$$

where $t \mapsto (X(t; x), R(t; x, r))$ is the solution of the ODE system

$$\dot{X}(t; x) = \Xi(t, X(t; x)), \quad \dot{R}(t; x, r) = 2\xi(t, X(t; x))R(t; x, r)$$

with initial conditions $X(0; x) = x$ and $R(0; x, r) = r$ is a lift of $\mu(t)$.

Note that we can solve the equation for R explicitly and obtain

$$R(t, x, r) = r \exp \left(2 \int_0^t \xi(s, X(s, x)) ds \right).$$

It is well-known that if Ξ fails to satisfy the regularity properties of Proposition 4.1 nothing guarantees uniqueness of the characteristics $t \mapsto (X(t), R(t))$ and formula (4.4) does not hold. To overcome this problem probability measures concentrated on entire trajectories in the underlying space \mathfrak{C}_Ω are introduced, see [Lis07] and [AGS05, Sect. 8.2].

More precisely, we call $\pi \in \mathcal{P}(C([0, 1]; \mathfrak{C}_\Omega))$ a dynamic plan if it is concentrated on absolutely continuous curves $\hat{z} \in \mathcal{A} := AC^2([0, 1]; (\mathfrak{C}_\Omega, d_\mathfrak{C}))$ and if it satisfies

$$\int_{\mathcal{A}} \left(\int_0^1 |\dot{\hat{z}}|_{\mathfrak{C}}(t)^2 dt \right) d\pi(\hat{z}) < \infty$$

with $|\dot{\hat{z}}|_{\mathfrak{C}}$ denoting the metric derivative with respect to the cone distance $d_\mathfrak{C}$, see (4.2). Note that any continuous curve $t \mapsto \hat{z}(t) = [\hat{x}(t), \hat{r}(t)]$ with $t \in [0, 1]$ satisfies $\hat{r} \in C([0, 1])$ with values in $[0, \infty[$. Thus, the set $O_{\hat{r}} = \hat{r}^{-1}(]0, \infty[) \subset [0, 1]$ is open and the restriction of \hat{x} to $O_{\hat{r}}$ is also continuous. The following lemma gives a characterization of the absolutely continuous curves in \mathfrak{C}_Ω and their metric derivative. It is proven in [LMS15].

Lemma 4.2 *A curve $t \mapsto \hat{z}(t) = [\hat{x}(t), \hat{r}(t)] \in \mathfrak{C}_\Omega$ satisfies $\hat{z} \in AC^p([0, 1]; \mathfrak{C}_\Omega)$ if and only if*

$$\dot{r} \in L^p(0, 1) \quad \text{and} \quad \hat{r}|_{\dot{\hat{x}}} \in L^p(O_{\hat{r}}) \quad \text{for } O_{\hat{r}} := \hat{r}^{-1}(]0, \infty[).$$

In particular, the metric time derivative is given via

$$|\dot{\hat{z}}|_{\mathfrak{C}}(t)^2 = \dot{r}(t)^2 + \hat{r}(t)^2 |\dot{\hat{x}}(t)|^2 \quad \text{for } t \in O_{\hat{r}} \quad \text{and} \quad |\dot{\hat{z}}|_{\mathfrak{C}}(t) = 0 \quad \text{otherwise.}$$

For $t \in [0, 1]$ we denote by $e_t : C([0, 1]; \mathfrak{C}_\Omega) \rightarrow \mathfrak{C}_\Omega$ the evaluation map given for $\hat{z} \in C([0, 1]; \mathfrak{C}_\Omega)$ by $e_t(\hat{z}) = \hat{z}(t)$. With a dynamic plan $\pi \in \mathcal{P}(C([0, 1]; \mathfrak{C}_\Omega))$ we associate the curve $t \mapsto \lambda(t) := (e_t)_\# \pi$ which belongs to $AC^2([0, 1]; (\mathcal{M}(\mathfrak{C}_\Omega), W_\mathfrak{C}))$, see [Lis07, Thm. 4]. Moreover, from the 1-Lipschitz continuity of the projection $\mathfrak{P} : \mathcal{M}(\mathfrak{C}_\Omega) \rightarrow \mathcal{M}(\Omega)$ it follows that the curve $t \mapsto \mu(t) := \mathfrak{P}\lambda(t)$ belongs to $AC^2([0, 1]; (\mathcal{M}(\Omega), \mathbf{H}\mathbf{K}))$ and the metric derivative of μ with respect to $\mathbf{H}\mathbf{K}$ satisfies

$$|\dot{\mu}|_{\mathbf{H}\mathbf{K}}(t)^2 \leq \int_{\mathcal{A}} |\dot{\hat{z}}|_{\mathfrak{C}}(t)^2 d\pi(\hat{z}). \quad (4.5)$$

The following theorem shows that for every absolutely continuous curve in $(\mathcal{M}(\Omega), \mathbf{H}\mathbf{K})$ a dynamic plan π exists such that μ is induced by π in the above sense and equality holds in (4.5). The proof is based on an extension of [Lis07, Thm. 5] and can be found in [LMS15, Thm. 8.4].

Theorem 4.3 *Let $\mu \in AC^2([0, 1]; (\mathcal{M}(\Omega), \mathbf{H}\mathbf{K}))$ be given. Then, there exists a dynamic plan $\pi \in \mathcal{P}(C([0, 1]; \mathfrak{C}_\Omega))$ such that $\mu(t) = \mathfrak{P}((e_t)_\# \pi)$ and*

$$|\dot{\mu}|_{\mathbf{H}\mathbf{K}}(t)^2 = \int_{\mathcal{A}} |\dot{\hat{z}}|_{\mathfrak{C}}(t)^2 d\pi(\hat{z}) \quad \text{for a.a. } t \in [0, 1]. \quad (4.6)$$

Using this result we can also show that all geodesic curves for the Hellinger-Kantorovich distance are given by projections of geodesic curves in $\mathcal{M}_2(\mathfrak{C}_\Omega)$, i.e. all geodesic curves have the representation (3.21).

Corollary 4.4 (Representation of all geodesic curves) *Let $[0, 1] \ni s \mapsto \mu(s)$ be a geodesic curve and π the dynamic plan from Theorem 4.3. Then, $s \mapsto \lambda(s) = (e_s)_\# \pi$ is a geodesic curve in $\mathcal{P}_2(\mathfrak{C}_\Omega)$ with respect to $W_\mathfrak{C}$.*

In particular, all geodesic curves in $(\mathcal{M}(\Omega), \mathbf{HK})$ are given by an optimal plan γ for optimal lifts of $\mu(0)$ and $\mu(1)$ in the form (3.21).

Proof: For $0 \leq s < t \leq 1$, we have the elementary estimates

$$\begin{aligned} W_\mathfrak{C}(\lambda(s), \lambda(t))^2 &\leq \int_{\mathfrak{C}_\Omega^2} d_\mathfrak{C}(z_0, z_1)^2 d(e_s, e_t)_\# \pi = \int_{\mathcal{A}} d_\mathfrak{C}(\widehat{z}(s), \widehat{z}(t))^2 d\pi \\ &\leq \int_{\mathcal{A}} \left(\int_s^t |\dot{\widehat{z}}|_\mathfrak{C} dr \right)^2 d\pi \leq (t-s) \int_s^t \int_{\mathcal{A}} |\dot{\widehat{z}}|_\mathfrak{C}^2 d\pi dr, \end{aligned}$$

where we have used Hölder's inequality. Since μ is a geodesic curve, we have $|\dot{\mu}|_{\mathbf{HK}} \equiv \mathbf{HK}(\mu(0), \mu(1))$ and hence, with (4.6) we have

$$W_\mathfrak{C}(\lambda(s), \lambda(t)) \leq (t-s) \mathbf{HK}(\mu(0), \mu(1)) \leq (t-s) W_\mathfrak{C}(\lambda(0), \lambda(1)).$$

Arguing as in the proof of Theorem 3.11 shows that $s \mapsto \lambda(s)$ is a geodesic curve. In particular, all inequalities above are equalities.

From the dynamic plan π we immediately find the optimal transport plan $\gamma := ((e_0), (e_1))_\# \pi$ between the optimal lifts $\lambda(0)$ and $\lambda(1)$, such that $\mu(s) = \mathfrak{P}\lambda(s)$. ■

The following theorem shows that for every curve $\mu \in \text{AC}^2(0, 1; (\mathcal{M}(\Omega), \mathbf{HK}))$ we can find a vector and a scalar field Ξ and ξ such that the continuity equation in (4.3) is satisfied. Moreover, the L^2 -norm of (Ξ, ξ) with respect to $\mu(t)$ provides a lower bound for the metric time derivative of μ .

Theorem 4.5 *Let $\mu \in \text{AC}^2([0, 1]; (\mathcal{M}(\Omega), \mathbf{HK}))$ be given. Then, there exists a Borel vector field $(\Xi, \xi) : [0, 1] \times \Omega \rightarrow \mathbb{R}^{d+1}$ such that the continuity equation (4.3) is satisfied and*

$$\int_{\Omega} \left[|\Xi(t, x)|^2 + 4|\xi(t, x)|^2 \right] d\mu(t) \leq |\dot{\mu}|_{\mathbf{HK}}(t)^2 \quad \text{for a.e. } t \in [0, 1].$$

Proof: Let $\pi \in \mathcal{P}(C([0, 1]; \mathfrak{C}_\Omega))$ be a dynamic plan representing μ according to Theorem 4.3. We denote the lift $\lambda(t) = (e_t)_\# \pi$. Due to the Disintegration Theorem, see [AGS05, Thm. 5.3.1], there exists a family of probability measures $\pi_z(t) \in \mathcal{P}(C([0, 1]; \mathfrak{C}_\Omega))$ for λ -a.e. $z \in \mathfrak{C}_\Omega$ and each $t \in [0, 1]$. Moreover, $\pi_z(t)$ is concentrated on the subset $\mathcal{A}_z(t) := \{\widehat{z} \in \mathcal{A} : \widehat{z}(t) = z\}$ and for every $F \in L^1(C([0, 1]; \mathfrak{C}_\Omega); \pi)$ we have

$$\int_{\mathcal{A}} F(\widehat{z}) d\pi(\widehat{z}) = \int_{\mathfrak{C}_\Omega} \left(\int_{\mathcal{A}_z(t)} F(\widehat{z}) d\pi_z(t) \right) d\lambda(t).$$

For $t \in [0, 1]$ and $z = [x, r] \in \mathfrak{C}_\Omega \setminus \{\mathfrak{o}\}$ we define the vector fields

$$\widetilde{\Xi}(t, z) = \int_{\mathcal{A}_z(t)} \dot{\widehat{x}}(t) d\pi_z(t) \quad \text{and} \quad \widetilde{\xi}(t, z) = \frac{1}{2} \int_{\mathcal{A}_z(t)} \frac{\dot{\widehat{r}}(t)}{\widehat{r}(r)} d\pi_z(t),$$

while for $z = \mathbf{o}$ we set $\tilde{\xi}(t, z) = \tilde{\Xi}(t, z) = 0$. Due to Jensen's inequality we have the estimate

$$\begin{aligned} \int_{\mathfrak{C}_\Omega} \left[|\tilde{\Xi}(t, z)|^2 + 4|\tilde{\xi}(t, z)|^2 \right] r^2 d\lambda(t) &\leq \int_{\mathfrak{C}_\Omega} \int_{\mathcal{A}_z(t)} \left[\hat{r}(t)^2 |\dot{\hat{x}}(t)|^2 + |\dot{\hat{r}}(t)|^2 \right] d\pi_z(t) d\lambda(t) \\ &= \int_{\mathcal{A}} |\dot{\hat{z}}|_{\mathfrak{C}}(t)^2 d\boldsymbol{\pi} = |\dot{\mu}|_{\mathbf{HK}}(t)^2. \end{aligned}$$

To obtain vector fields Ξ and ξ on Ω we employ the Disintegration Theorem for $r^2\lambda$ and $\mu = \Pi_{\#}(r^2\lambda)$ to obtain a family of probability measures ν_x concentrated on $[0, \infty[$ and such that $r^2\lambda = \nu_x\mu$. Using again Jensen's inequality we easily check that the fields $\Xi(t, x) = \int_{[0, \infty[} \tilde{\Xi}(t, z) d\nu_x$ and $\xi(t, x) = \int_{[0, \infty[} \tilde{\xi}(t, z) d\nu_x$ satisfy

$$\int_{\Omega} \left[|\Xi(t, x)|^2 + 4|\xi(t, x)|^2 \right] d\mu(t) \leq |\dot{\mu}|_{\mathbf{HK}}(t)^2 \quad \text{for a.e. } t \in [0, 1].$$

It remains to show that the continuity equation (4.3) is satisfied. For this, we choose a test function of the form $\varphi(x, t) = \eta(x)\psi(t)$ with η and ψ Lipschitz and bounded with compact support in Ω and $]0, 1[$, respectively. We compute

$$\begin{aligned} \int_0^1 \int_{\Omega} \eta(x) \dot{\psi}(t) d\mu(t) dt &= \int_{\mathcal{A}} \int_0^1 \dot{\psi}(t) \eta(\hat{x}(t)) \hat{r}(t)^2 dt d\boldsymbol{\pi} \\ &= - \int_{\mathcal{A}} \int_0^1 \psi(t) \left[\nabla \eta(x(t)) \dot{x}(t) + 4\eta(x(t)) \frac{\dot{r}(t)}{2\hat{r}(t)} \right] r(t)^2 dt d\boldsymbol{\pi} \\ &= - \int_{\mathfrak{C}_\Omega} \int_0^1 \psi(t) [\nabla \eta(x) \cdot \tilde{\Xi}(t, z) + \eta(x) \tilde{\xi}(t, z)] r^2 d\lambda(t) dt \\ &= - \int_{\Omega} \int_0^1 \psi(t) [\nabla \eta(x) \cdot \Xi(t, x) + \eta(x) \xi(t, x)] d\mu(t) dt. \end{aligned}$$

Thus, the continuity equation is satisfied in the distributional sense. ■

Next, we show the reverse implication.

Theorem 4.6 *Let $t \mapsto \mu(t)$ be a narrowly continuous curve in $\mathcal{M}(\Omega)$ and suppose that there exists a Borel vector field $\Xi : [0, 1] \times \Omega \rightarrow \mathbb{R}^d$ and a scalar field $\xi : [0, 1] \times \Omega \rightarrow \mathbb{R}$ satisfying $(\Xi(t), \xi(t)) \in L^2(d\mu(t); \mathbb{R}^{d+1})$*

$$\int_0^1 \int_{\Omega} \left(|\Xi(t, x)|^2 + 4|\xi(t, x)|^2 \right) d\mu(t) dt < \infty$$

such that the continuity equation (4.3) is satisfied. Then, $\mu \in \text{AC}^2([0, 1]; (\mathcal{M}(\Omega), \mathbf{HK}))$ and

$$|\dot{\mu}|_{\mathbf{HK}}(t)^2 \leq \int_{\Omega} \left(|\Xi(t, x)|^2 + 4|\xi(t, x)|^2 \right) d\mu(t) \quad \text{for a.e. } t \in]0, 1[. \quad (4.7)$$

Proof: Let μ , Ξ , and ξ be given as in the statement of the theorem. Due to Lemma 3.10 in [Man07] for $\varepsilon > 0$ we can obtain sufficiently smooth approximations μ_ε , Ξ_ε , and ξ_ε satisfying the continuity equation (4.3) and converging in a suitable sense to μ , Ξ , and ξ , respectively.

Moreover, μ_ε , Ξ_ε , and ξ_ε satisfy for any convex, nondecreasing function $\psi : [0, \infty[\rightarrow [0, \infty[$ we have the uniform estimates

$$\begin{aligned} \int_{\Omega} \psi(|\Xi_\varepsilon(t, x)|) d\mu_\varepsilon(t) &\leq \int_{\Omega} \psi(|\Xi(t, x)|) d\mu(t) \quad \text{and} \\ \int_{\Omega} \psi(|\xi_\varepsilon(t, x)|) d\mu_\varepsilon(t) &\leq \int_{\Omega} \psi(|\xi(t, x)|) d\mu(t) \end{aligned} \quad (4.8)$$

Applying the representation result in Proposition 4.1 for μ_ε , Ξ_ε , and ξ_ε we obtain the formula

$$\mu_\varepsilon(t) = \mathfrak{P}\lambda_\varepsilon(t), \quad \lambda_\varepsilon(t) = [X_\varepsilon(t; \cdot), R_\varepsilon(t; \cdot)]_{\#} \lambda_\varepsilon(0), \quad (4.9)$$

where $\lambda_\varepsilon(0)$ is a lift of $\mu_\varepsilon(0)$ and X_ε and R_ε are the maximal solutions of

$$\dot{X}_\varepsilon(t; x) = \Xi_\varepsilon(t, X_\varepsilon(t; x)), \quad \dot{R}_\varepsilon(t; x, r) = 2\xi(t, X_\varepsilon(t; x))R_\varepsilon(t; x, r) \quad (4.10)$$

subject to the initial conditions $X_\varepsilon(0; x) = x$ and $R_\varepsilon(0; x, r) = r$. With this we define the map $\Phi_\varepsilon : \mathfrak{C}_\Omega \rightarrow C([0, 1]; \mathfrak{C}_\Omega)$ via

$$\Phi_\varepsilon([x, r]) = (t \mapsto [X_\varepsilon(t; x), R_\varepsilon(t; x, r)]),$$

and introduce the dynamic plan $\pi_\varepsilon \in \mathcal{M}(C([0, 1]; \mathfrak{C}_\Omega))$ as $\pi_\varepsilon = (\Phi_\varepsilon)_{\#} \lambda_\varepsilon(0)$.

We aim to show that the sequence π_ε is tight such that we can find a subsequence (not relabeled) that is narrowly converging to a dynamic plan π . Indeed, using Lemma 4.2, (4.10) and the representation formula (4.9) we have for $0 \leq t_0 < t_1 \leq 1$ that

$$\begin{aligned} \int_{\mathcal{A}} \int_{t_0}^{t_1} |\dot{\hat{z}}|_{\mathfrak{C}_\Omega}^2(t) dt d\pi_\varepsilon(\hat{z}) &= \int_{\mathfrak{C}_\Omega} \int_{t_0}^{t_1} \left\{ R_\varepsilon(t; x, r)^2 |\dot{X}_\varepsilon(t, x)|^2 + \dot{R}_\varepsilon(t; x, r)^2 \right\} dt d\lambda_\varepsilon(0) \\ &= \int_{\mathfrak{C}_\Omega} \int_{t_0}^{t_1} \left\{ |\Xi_\varepsilon(t, X_\varepsilon)|^2 + 4|\xi_\varepsilon(t, X_\varepsilon)|^2 \right\} R_\varepsilon^2 dt d\lambda_\varepsilon(0) \\ &= \int_{t_0}^{t_1} \int_{\Omega} \left\{ |\Xi_\varepsilon(t, x)|^2 + 4|\xi_\varepsilon(t, x)|^2 \right\} d\mu_\varepsilon(t) dt \\ &\leq \int_{t_0}^{t_1} \int_{\Omega} \left\{ |\Xi(t, x)|^2 + 4|\xi(t, x)|^2 \right\} d\mu(t) dt < \infty. \end{aligned}$$

Since the functional $\hat{z} \mapsto \int_0^1 |\dot{\hat{z}}|_{\mathfrak{C}_\Omega}^2 dt$ has compact sublevels in $\{\hat{z} \in C([0, T]; \mathfrak{C}_\Omega) \mid \hat{z}(0) = [x, r]\}$ we have shown the tightness of π_ε and we can extract a subsequence narrowly converging to a limit π in $\mathcal{M}(C([0, 1]; \mathfrak{C}_\Omega))$. Moreover, due to the lower semicontinuity of $\hat{z} \mapsto \int_0^1 |\dot{\hat{z}}|_{\mathfrak{C}_\Omega}^2 dt$ we immediately obtain

$$\int_{\mathcal{A}} \int_{t_0}^{t_1} |\dot{\hat{z}}|_{\mathfrak{C}_\Omega}^2 dt d\pi(\hat{z}) < \infty.$$

Hence, π is concentrated on absolutely continuous curves.

It remains to show that π is a dynamic plan for μ , i.e., $\mu(t) = \mathfrak{P}((e_t)_{\#} \pi)$ from which (4.7) follows. We easily show that due to the construction of π_ε we have that $\mathfrak{P}((e_t)_{\#} \pi_\varepsilon) = \mu_\varepsilon(t)$. The claim now follows from the continuity of the map $\pi \mapsto \mathfrak{P}((e_t)_{\#} \pi)$ with respect to the weak convergence in $\mathcal{M}(C([0, 1]; \mathfrak{C}_\Omega))$. \blacksquare

Finally, we can prove the equivalence of the dynamical and the transport plan formulation of the Hellinger–Kantorovich distance.

Proof of Theorem 3.6(v). We want to show that for all $\mu_0, \mu_1 \in \mathcal{M}(\Omega)$ we have

$$\mathbf{HK}(\mu_0, \mu_1) = \mathbf{D}_{1,4}(\mu_0, \mu_1).$$

Due to Theorem 4.6 we have for any curve $t \mapsto \mu(t)$ connecting μ_0 and μ_1 and satisfying the continuity equation (4.3) for a vector and scalar field Ξ and ξ , respectively, the upper estimate

$$\mathbf{HK}(\mu_0, \mu_1)^2 \leq \int_0^1 |\dot{\mu}|_{\mathbf{HK}}^2(t) dt \leq \int_0^1 \int_{\Omega} \left\{ |\Xi(t, x)|^2 + 4\xi(t, x)^2 \right\} d\mu(t) dt.$$

Thus, by minimizing over all absolutely continuous curves $t \mapsto \mu(t)$ connecting μ_0 and μ_1 we have $\mathbf{HK}(\mu_0, \mu_1) \leq \mathbf{D}_{1,4}(\mu_0, \mu_1)$.

To show that equality holds, we consider a geodesic curve $s \mapsto \mu(s)$, which is obviously absolutely continuous with respect to \mathbf{HK} and we have $|\dot{\mu}|_{\mathbf{HK}} \equiv \mathbf{HK}(\mu_0, \mu_1)$. By Theorem 4.5 we then have

$$\mathbf{HK}(\mu_0, \mu_1)^2 = \int_0^1 |\dot{\mu}|_{\mathbf{HK}}^2(t) dt \geq \int_0^1 \int_{\Omega} \left\{ |\Xi(t, x)|^2 + 4\xi(t, x)^2 \right\} d\mu(t) dt.$$

Hence, we have proven Theorem 3.6(v). ■

5 Geodesic curves for \mathbf{HK}

In this section, we provide some general results on geodesics as well as a few illuminating examples. Section 5.1 deals with the geodesic Λ -convexity of functionals $\mathcal{F} : \mathcal{M}(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$. In particular, we show that the linear functional $\mathcal{F}_{\Phi} : \mu \mapsto \int_{\Omega} \Phi(x) d\mu(x)$ is geodesically Λ -convex if and only if the function $[x, r] \mapsto r^2 \Phi(x)$ is Λ -convex on $(\mathfrak{C}_{\Omega}, d_{\mathfrak{C}})$. In Section 5.2, we return to the problem of moving the measure $\mu_0 = a_0 \delta_{y_0}$ to $\mu_1 = a_1 \delta_{y_1}$ and show that in the case $|y_1 - y_0| = \pi/2$ there is an infinite set of geodesic curves. Moreover, we show that all these curves are indeed solutions of the formally derived equation

$$\frac{d}{ds} \mu + \operatorname{div}(\mu \nabla \xi) = 4\xi \mu, \quad \frac{d}{ds} \xi + \frac{1}{2} |\nabla \xi|^2 + 2\xi^2 = 0, \quad (5.1)$$

where ξ is in fact the same for all geodesic connections. In Section 5.3 we discuss geodesic curves that are induced by dilation of measures. Section 5.4 discusses how the geodesic curve connecting $\mu_0 = \chi_{[0,1]}$ and $\mu_1 = \chi_{[2,3]}$ can be constructed.

Finally, Section 5.6 shows that $(\mathcal{M}(\Omega), \mathbf{HK})$ is not a positively curved (PC) space in the sense of Alexandrov (cf. [AGS05, Sect. 12.3]) if Ω is two-dimensional.

5.1 Geodesic Λ -convexity for some functionals

Here we give some first results of geodesic Λ -convexity for functionals $\mathcal{F} : \mathcal{M}(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$, which is defined via

$$\begin{aligned} & \forall \text{ geod. curves } \mu : [0, 1] \rightarrow \mathcal{M}(\Omega) : \\ & \mathcal{F}(\mu(s)) \leq (1-s)\mathcal{F}(\mu(0)) + s\mathcal{F}(\mu(1)) - \Lambda \frac{s(1-s)}{2} \mathbf{HK}(\mu(0), \mu(1))^2. \end{aligned} \quad (5.2)$$

We first provide an exact characterizations of Λ for linear functionals, then give some preliminary results and conjectures for nonlinear functionals.

It is well-known that in the case of the Wasserstein distance, geodesic Λ -convexity of functionals of the form $\mathcal{F}_\Phi(\mu) = \int_\Omega \Phi(x) d\mu(x)$ is satisfied if and only if $x \mapsto \Phi(x)$ is Λ -convex, see [AGS05, Sect. 9.3]. Hence, it is natural to ask whether the same can be said for the Hellinger–Kantorovich distance \mathbf{HK} and geodesics in the cone space.

We start with a very easy relation for the total mass along the geodesic curves. It turns out that the mass depends on the parameter convex and quadratically.

Proposition 5.1 *Consider a geodesic curve $[0, 1] \ni s \mapsto \mu(s)$ given by (3.21) and set*

$$m(s) := |\mu|(s) = \int_\Omega d\mu(s).$$

Then, we have

$$m(s) = (1-s)^2 m(0) + s^2 m(1) + 2s(1-s)m_* \quad (5.3a)$$

$$\text{with } m_* := \int_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} r_0 r_1 \cos_\pi(|x_1 - x_0|) d\gamma([x_0, r_0], [x_1, r_1]),$$

$$\frac{m(0)m(1)}{m(0)+m(1)} \leq (1-s)^2 m(0) + s^2 m(1) \leq m(s) \leq \left((1-s)\sqrt{m(0)} + s\sqrt{m(1)} \right)^2 \quad (5.3b)$$

for all $s \in [0, 1]$. Moreover, $m''(s) = 2\mathbf{HK}(\mu_0, \mu_1)^2 \geq 0$ which implies

$$m(s) = (1-s)m(0) + sm(1) - s(1-s)\mathbf{HK}(\mu_0, \mu_1)^2. \quad (5.4)$$

Proof: Using the definition of $\mu(s)$ via the projection and $Z(s; \cdot, \cdot)$ in (3.13) we have

$$m(s) = \int_\Omega d\mu(s) = \int_{\mathfrak{C}_\Omega} r^2 d\lambda(s) = \int_{\mathfrak{C}_\Omega} r^2 d(Z(s; \cdot, \cdot)_\# \gamma) = \int_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} R(s; z_0, z_1)^2 d\gamma(z_0, z_1).$$

Now, we can use the explicit quadratic structure of R^2 given in (3.13) we find the quadratic formula (5.3a).

For estimate (5.3b) we use the fact that $\cos_\pi(|x_1 - x_0|)$ takes values only in the interval $[0, 1]$ on the support of γ , see Proposition 3.4(c). By the Cauchy-Schwarz estimate we have $0 \leq m_* \leq \sqrt{m(0)m(1)}$, which implies the estimates.

Obviously, we have $m''(s) = 2(m(0) + m(1) - 2m_*)$, and comparing to the characterization (3.20) we find $m''(s) = 2\mathbf{HK}(\mu_0, \mu_1)^2$ as desired. \blacksquare

Next, we consider the linear functional $\mathcal{F}_\Phi(\mu) = \int_\Omega \Phi(x) d\mu(x)$ with $\Phi \in C^0(\Omega)$.

Proposition 5.2 *Let $\Phi \in C^0(\Omega)$ be given and define $\tilde{\Phi}([x, r]) = r^2\Phi(x)$. Then the functional $\mathcal{F}_\Phi : \mathcal{M}(\Omega) \rightarrow \mathbb{R}$ is Λ -convex along $[0, 1] \ni s \mapsto \mu(s)$ given by (3.21) if and only if $\tilde{\Phi} : \mathfrak{C}_\Omega \rightarrow \mathbb{R}$ is Λ -convex.*

Proof: Assume that $\tilde{\Phi} : \mathfrak{C}_\Omega \rightarrow \mathbb{R}$ is Λ -convex. We use the definition of $s \mapsto \mu(s)$ in (3.21) to find

$$\mathcal{F}_\Phi(\mu(s)) = \int_\Omega r^2 \Phi(x) d\lambda(s) = \int_{\mathfrak{C}_\Omega} \tilde{\Phi}(Z(s; \cdot, \cdot)) d\gamma,$$

where $\gamma \in \mathcal{M}_2(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega)$ is an optimal plan for $\mu_0 = \mu(0)$ and $\mu_1 = \mu(1)$. Thus, with the convexity of $\tilde{\Phi}$ and the optimality of γ we find

$$\mathcal{F}_\Phi(\mu(s)) \leq (1-s)\mathcal{F}_\Phi(\mu_0) + s\mathcal{F}_\Phi(\mu_1) - \frac{\Lambda}{2}s(1-s)\mathbf{HK}(\mu_0, \mu_1)^2.$$

Conversely, if \mathcal{F}_Φ is Λ -convex on $\mathcal{M}(\Omega)$ we can consider geodesic curves for two Dirac measures $\mu_0 = a_0\delta_{x_0}$ and $\mu_1 = a_1\delta_{x_1}$ to obtain convexity of $\tilde{\Phi}$ along geodesics in \mathfrak{C}_Ω . However, as transport above the threshold $\pi/2$ is not optimal, this excludes geodesic curves in \mathfrak{C}_Ω for distances $\pi/2 < |x_0 - x_1| < \pi$. In this case, we note that we can always reduce this case to two overlapping geodesic curves for distances below $\pi/2$. ■

Finally, we provide some negative results for functionals that are geodesically Λ -convex for the Wasserstein–Kantorovich distance but not for the Hellinger–Kantorovich distance. A simple necessary condition is obtained by realizing that for all $\mu_1 \in \mathcal{M}(\Omega)$ the Hellinger geodesic

$$\mu^H(s) := s^2 \mu_1$$

is also the unique geodesic in $(\mathcal{M}(\Omega), \mathbf{HK})$ connecting $\mu_0 = 0$ and μ_1 . Indeed, this easily follows from the fact that the possible lifts of μ_0 are given by $\alpha\delta_{\mathfrak{o}} \in \mathcal{M}_2(\mathfrak{C}_\Omega)$ with $\alpha \geq 0$. However, geodesics in $(\mathfrak{C}_\Omega, \mathbf{d}_c)$ connecting \mathfrak{o} and $z_1 = [x, r]$ are simply given by $z(s) = [x, sr]$.

Applying this to Boltzmann’s logarithmic entropy $\mathcal{E} : \mu \mapsto \int_\Omega F_B(d\mu/dx) dx$, we see that it is not geodesically Λ -convex with respect to \mathbf{HK} . For this, consider the geodesic $\mu(s) = s^2 u dx$, where $u \in L^2(\Omega)$, $u \geq 0$, and $|\mu| = \int_\Omega u dx > 0$ to find the relation

$$\mathcal{E}(\mu(s)) = s^2 \mathcal{E}(u dx) + (1-s^2) \int_\Omega 1 dx + 2s^2 \log s \int_\Omega u dx.$$

Clearly, the last term destroys geodesic Λ -convexity. Similarly, for $p \in]0, 1[\cup]1, \infty[$ we may look at functionals of the form

$$\mathcal{E}_p(\mu) = \int_\Omega \frac{1}{p-1} \left(\frac{d\mu}{dx} \right)^p dx. \quad (5.5)$$

Along the geodesics $\mu(s) = s^2 u dx$ we obtain $e_p(s) := \mathcal{E}_p(\mu(s)) = s^{2p} \mathcal{E}_p(\mu(1))$. For $p \in]1/2, 1[$ we conclude $e_p''(s) \rightarrow -\infty$ for $s \downarrow 0$ due to $e_p(1) < 0$. Hence, for these p the functional \mathcal{E}_p is not geodesically Λ -convex for any $\lambda \in \mathbb{R}$ with respect to \mathbf{HK} .

The following remark supports the conjecture that the functional \mathcal{E}_p is geodesically convex on $(\mathcal{M}(\Omega), \mathbf{HK})$, or more generally on $(\mathcal{M}(\Omega), \mathbf{D}_{\alpha, \beta})$ for all $p > 1$. It is based on the formal differential calculus developed in [LiM13], which was in fact the stimulus of

this work. If this is the case one may consider the geodesically Λ -convex gradient system $(\mathcal{M}(\Omega), \mathcal{E}_p + \mathcal{F}_\Phi, \mathbf{D}_{\alpha, \beta})$, which corresponds to the partial differential equation

$$\partial_t u = -\mathbb{K}_{\alpha, \beta}(u) \mathbf{D}(\mathcal{E}_p + \mathcal{F}_\Phi)(u) = \alpha \left(\Delta(u^p) + \operatorname{div}(u \nabla \Phi) \right) - \beta \left(\Phi u + \frac{u^p}{p-1} \right),$$

complemented by the no-flux boundary conditions $\nabla \left(\frac{p}{p-1} u^{p-1} + \Phi \right) \cdot \nu = 0$. Note that this equation always has the solution $u \equiv 0$, which is different from the unique minimizer u_{\min} of $\mathcal{E}_p + \mathcal{F}_\Phi$, if Φ attains negative values somewhere. Indeed, we have $u_{\min} := \left(\frac{p-1}{p} \max\{0, -\Phi\} \right)^{1/(p-1)}$. We refer to [PQV14, Eqn. (2.1)] for an application for modeling of tumor growth.

Remark 5.3 (Geodesic convexity via Eulerian calculus) *Following ideas in [OtW05, DaS08] a formal calculus for reaction-diffusion systems was developed in [LiM13]. The idea is to characterize the geodesic Λ -convexity of $\mathcal{E}(u \, dx) = \int_\Omega E(u(x)) \, dx$ on $(\mathcal{M}(\Omega), \mathbf{D}_{\alpha, \beta})$ by calculating the quadratic form $M(u, \cdot)$ generated by the contravariant Hessian of \mathcal{E} :*

$$M(u, \xi) = \langle \xi, \mathbf{D}\mathbf{V}(u) \mathbb{K}_{\alpha, \beta}(u) \xi \rangle - \frac{1}{2} \mathbf{D}_u \langle \xi, \mathbb{K}_{\alpha, \beta}(u) \xi \rangle [\mathbf{V}(u)] \text{ with } \mathbf{V}(u) = \mathbb{K}_{\alpha, \beta}(u) \mathbf{D}\mathcal{E}(u).$$

Then, one needs to show the estimate $M(u, \xi) \geq \Lambda \langle \xi, \mathbb{K}_{\alpha, \beta}(u) \xi \rangle$.

Following the methods in [LiM13, Sect. 4], for $u \in C_c^0(\Omega)$ and smooth ξ we obtain

$$\begin{aligned} M(u, \xi) = \int_\Omega & \left(\alpha^2 \left((A(u) - H(u)) (\Delta \xi)^2 + H(u) |\mathbf{D}^2 \xi|^2 \right) \right. \\ & \left. + \alpha \beta \left(B_1(u) |\nabla \xi|^2 + B_2(u) \xi \Delta \xi \right) + \beta^2 B_3(u) \xi^2 \right) dx, \end{aligned}$$

$$\text{where } A(u) = u^2 E''(u), \quad H(u) = u E'(u) - E(u), \quad B_1(u) = \frac{3u}{2} E'(u) - E(u),$$

$$B_2(u) = -2u^2 E''(u) + u E'(u) - E(u), \quad B_3(u) = u^2 E''(u) + \frac{u}{2} E'(u).$$

For the special case $E(u) = u^p/(p-1)$ with $p > 1$ we find the relation

$$M_p(u, \xi) = \int_\Omega \left[\alpha^2 \left((p-1) (\Delta \xi)^2 + |\mathbf{D}^2 \xi|^2 \right) + \alpha \beta \left(\frac{3p-2}{2p-2} |\nabla \xi|^2 - (2p-1) \xi \Delta \xi \right) + \beta^2 \frac{2p^2-p}{2p-2} \xi^2 \right] u^p \, dx,$$

which is nonnegative, because the mixed term $\xi \Delta \xi$ can be estimated via

$$-\alpha \beta (2p-1) \xi \Delta \xi \geq -\alpha^2 (p-1) (\Delta \xi)^2 - \frac{(2p-1)^2}{4(p-1)} \beta^2 \xi^2.$$

Thus, the formal Eulerian calculus suggests that \mathcal{E}_p is geodesically convex with respect to \mathbb{H} for all $p > 1$. This investigation will be continued in subsequent work.

5.2 Geodesic connections for two Dirac measures

While for the characterization of the distance \mathbf{HK} the choice of the geodesics is not so relevant, we want to highlight that the set of geodesic connections between two measures can be very large. As shown in Section (3.4) and Corollary 4.4 all geodesic curves can be constructed from optimal couplings $\gamma \in \mathcal{M}_2(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega)$ in (3.16). However, many γ lead via the projection \mathfrak{P} to the same geodesic curve. Here we show that the set of geodesics may still form an infinite-dimensional convex set.

We treat the case of two Dirac measures $a_j \delta_{y_j}$. The case $|y_1 - y_0| \neq \pi/2$ is trivial, since only one geodesic connection exists. However, for the critical distance $|y_1 - y_0| = \pi/2$ an uncountable number of linearly independent connecting geodesics exists, such that the span of the convex set of all geodesics is infinite dimensional.

We consider $\mu_0 = a_0 \delta_{y_0}$ and $\mu_1 = a_1 \delta_{y_1}$ with $a_0, a_1 > 0$. As was shown before there is exactly one connecting geodesics if $|y_0 - y_1| \neq \pi/2$. Indeed, for $|y_0 - y_1| < \pi/2$ we have $\mu(s) = a(s) \delta_{x(s)}$ as discussed in Section 3.1. For $|y_0 - y_1| > \pi/2$ we have a pure Hellinger case with $\mu(s) = (1-s)^2 a_0 \delta_{y_0} + s^2 a_1 \delta_{y_1}$.

For the critical case $|y_0 - y_1| = \pi/2$ we have a huge set of possible geodesics, since we may consider all lifts $\hat{\lambda}_0, \hat{\lambda}_1 \in \mathcal{M}([0, \infty[)$ satisfying

$$a_j = \int_{[0, \infty[} r^2 d\hat{\lambda}_j(r) \text{ for } j = 1, 2 \text{ and } \hat{\lambda}_0([0, \infty[) = \hat{\lambda}_1([0, \infty[).$$

Now every coupling $\hat{\gamma} \in \Gamma(\hat{\lambda}_0, \hat{\lambda}_1) = \{ \hat{\gamma} \in \mathcal{M}([0, \infty[^2) \mid \Pi_{\#}^i \hat{\gamma} = \hat{\lambda}_i \}$ provides an optimal coupling. To see this, we set $\lambda_j = \delta_{y_j} \otimes \hat{\lambda}_j \in \mathcal{M}(\mathfrak{C}_\Omega)$ and $\gamma = \delta_{y_0} \otimes \delta_{y_1} \otimes \hat{\gamma} \in \mathcal{M}(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega)$. Since $|y_0 - y_1| = \pi/2$ implies $d_{\mathfrak{C}}([y_0, r_0], [y_1, r_1])^2 = r_0^2 + r_1^2$, we find

$$\begin{aligned} \int_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} d_{\mathfrak{C}}([x_0, r_0], [x_1, r_1])^2 d\gamma &= \int_{[0, \infty[^2} d_{\mathfrak{C}}([y_0, r_0], [y_1, r_1])^2 d\hat{\gamma} \\ &= \int_{[0, \infty[^2} (r_0^2 + r_1^2) d\hat{\gamma} = a_0 + a_1 = \mathbf{HK}(a_0 \delta_{y_0}, a_1 \delta_{y_1})^2. \end{aligned}$$

Now, geodesic curves can be constructed for every $\hat{\gamma}$ as defined in (3.21). Obviously the set of all possible pairs $(\hat{\lambda}_0, \hat{\lambda}_1)$ is convex and therefore also the set of all $\hat{\gamma} \in \Gamma(\hat{\lambda}_0, \hat{\lambda}_1)$. Hence, the set of all optimal $\hat{\gamma}$ is convex.

However, the mapping from $\hat{\gamma}$ to $\mu(\cdot)$ is not surjective, since there is a huge redundancy. Indeed, by the definition $\mu_s = \mathfrak{P} Z(s; \cdot, \cdot)_{\#} \gamma$ we have

$$\int_{\Omega} \psi(x) d\mu_s(x) = \int_{[0, \infty[^2} R(s, [y_0, r_0], [y_1, r_1])^2 \psi(X(s, [y_0, r_0], [y_1, r_1])) d\hat{\gamma}(r_0, r_1),$$

where, using $|y_0 - y_1| = \pi/2$, we have $R(s, [y_0, r_0], [y_1, r_1])^2 = (1-s)^2 r_0^2 + s^2 r_1^2$ and

$$X(s, [y_0, r_0], [y_1, r_1]) = (1-\rho(s))y_0 + \rho(s)y_1 \text{ with } \rho(s) = \frac{2}{\pi} \arccos \left[1 + \left(\frac{sr_1}{(1-s)r_0} \right)^2 \right]^{-1/2},$$

see (3.13). The observation is that the integrand can be written in the form $r_1^2 \Phi(s, r_0/r_1)$. In particular, for $r_1 > 0$ the two geodesics

$$\begin{aligned} \Lambda(s) &= \delta_{R(s; [y_0, r_0], [y_1, r_1])} \otimes \delta_{X(s; [y_0, r_0], [y_1, r_1])} \quad \text{and} \\ \tilde{\Lambda}(s) &= \delta_{r_1 R(s; [y_0, r_0/r_1], [y_1, 1])} \otimes \delta_{X(s; [y_0, r_0/r_1], [y_1, 1])} \end{aligned}$$

on \mathfrak{C}_Ω give rise, via the projection \mathfrak{P} , to the same geodesic on Ω given by

$$\mu(s) = R(s; [y_0, r_0], [y_1, r_1])^2 \delta_{X(s; [y_0, r_0], [y_1, r_1])}.$$

Thus, for a coupling $\gamma \in \Gamma(\delta_{y_0} \otimes \hat{\lambda}_0, \delta_{y_1} \otimes \hat{\lambda}_1)$ given by $\hat{\gamma} \in \Gamma(\hat{\lambda}_0, \hat{\lambda}_1)$ we can define the normalization $N_0\gamma \in \mathcal{M}(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega)$ with respect to $s = 0$ as follows ($N_1\gamma$ for $s = 1$ can be defined similarly):

$$\begin{aligned} \int_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} \Phi(z_0, z_1) dN_0\gamma &= \int_{]0, \infty[\times]0, \infty[} r_1^2 \Phi([y_0, r_0/r_1], [y_1, 1]) d\hat{\gamma}(r_0, r_1) \\ &\quad + \Phi(\mathfrak{o}, \mathfrak{o})b_{\mathfrak{o}, \mathfrak{o}} + \Phi([y_0, 1], \mathfrak{o})b_0 + \Phi(\mathfrak{o}, [y_1, 1])b_1, \end{aligned}$$

$$\text{where } b_{\mathfrak{o}, \mathfrak{o}} := \hat{\gamma}(\{(0, 0)\}), \quad b_0 := \int_{]0, \infty[} r_0^2 d\hat{\gamma}(r_0, 0), \quad b_1 := \int_{]0, \infty[} r_1^2 d\hat{\gamma}(0, r_1).$$

The terms involving b_j contain the trivial Hellinger terms, where mass is moved into or generated out of the tip \mathfrak{o} . Note that this mass can be concentrated to the fixed value $r_0 = 1$ or $r_1 = 1$, respectively. The second term gives the mass that simply stays in \mathfrak{o} . The interesting part is the first one, where still a measure $\mathbf{n}_0\hat{\gamma}$ survives:

$$\int_{]0, \infty[\times]0, \infty[} r_1^2 \Phi([y_0, r_0/r_1], [y_1, 1]) d\hat{\gamma}(r_0, r_1) =: \int_{]0, \infty[} \Phi([y_0, r], [y_1, 1]) d(\mathbf{n}_0\hat{\gamma})(r).$$

We will see below that $(\mathbf{n}_0\hat{\gamma})(dr)$ gives the mass that leaves y_0 with speed $1/r$.

It is easy to see that γ and $N_0\gamma$ generate the same geodesic curve $\mu(\cdot)$. In terms of $\mathbf{n}_0\hat{\gamma}$, we can now write the geodesic curve associated with γ in a simpler form, namely

$$\begin{aligned} \int_{\Omega} \psi(x) d\mu_s(x) &= \int_{]0, \infty[} ((1-s)^2 r^2 + s^2) \psi((1-\hat{\rho}(s, r))y_0 + \hat{\rho}(s, r)y_1) d(\mathbf{n}_0\hat{\gamma})(r) \\ &\quad + 0 + (1-s)^2 b_0 \psi(y_0) + s^2 b_1 \psi(y_1), \end{aligned}$$

where $\hat{\rho}(s, r) = \frac{2}{\pi} \arccos \left[1 + \left(\frac{s}{(1-s)r} \right)^2 \right]^{-1/2} \in]0, 1[$ for $s \in]0, 1[$ and $r > 0$ and

$$a_0 = b_0 + \int_{]0, \infty[} r^2 d(\mathbf{n}_0\hat{\gamma})(r) \quad \text{and} \quad a_1 = b_1 + \int_{]0, \infty[} d(\mathbf{n}_0\hat{\gamma})(r).$$

To simplify the further notation we now assume $\Omega = [-2, 2]$, $y_0 = 0$, and $y_1 = \pi/2$. By the definition of $\hat{\rho}$ we find $x = \hat{x}(r, s) := (1-\hat{\rho}(s, r))y_0 + \hat{\rho}(s, r)y_1$ if and only if $r = (s \cos x) / ((1-s) \sin x)$. Now differentiating $\int_{[0, \pi/2]} \psi(x) d\mu_s(x)$ with respect to s we find

$$\begin{aligned} \frac{d}{ds} \int_{[0, \pi/2]} \psi(x) d\mu_s(x) &= -2(1-s)b_0\psi(0) + 2sb_1\psi(\pi/2) \\ &\quad + \int_{]0, \infty[} \left(2(s-(1-s)r^2)\psi(\hat{x}(r, s)) + ((1-s)^2 r^2 + s^2)\psi'(\hat{x}(r, s))\partial_s \hat{x}(r, s) \right) d(\mathbf{n}_0\hat{\gamma}). \end{aligned}$$

Defining the function ξ and Ξ explicitly via

$$\xi(s, x) = \frac{(\sin x)^2 - s}{2s(1-s)} \quad \text{and} \quad \Xi(s, x) = \partial_x \xi(s, x) \tag{5.6}$$

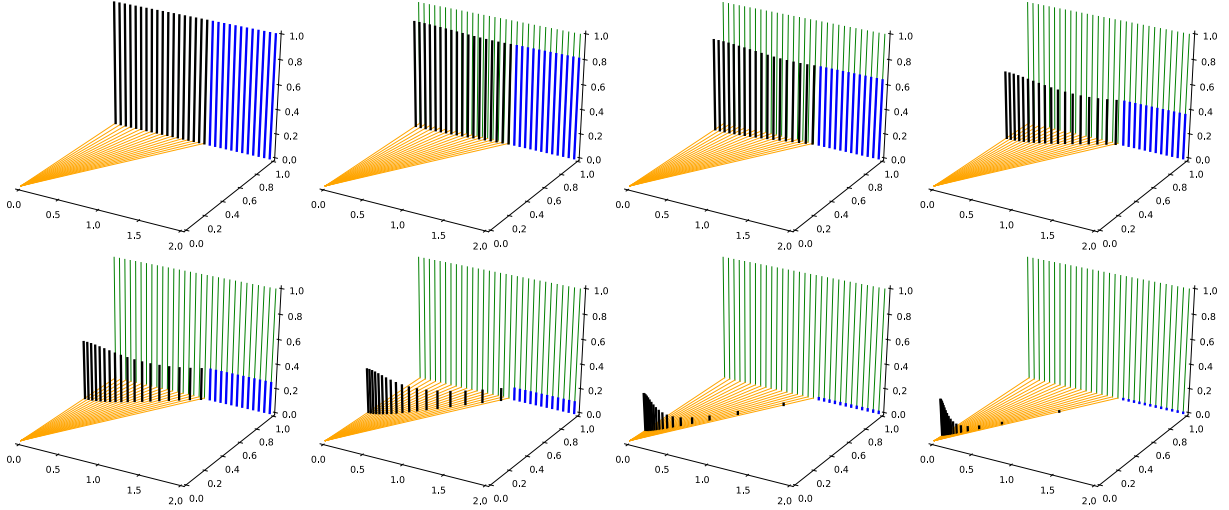


Figure 5: Geodesic curve $s \mapsto \mu(s)$ connecting superposition of Dirac masses and single Dirac measure in Example 5.4(i) for different values of s . Here green denotes the measure μ_1 for $N = 30$ while the blue and the black parts correspond to the parts of $\mu(s)$ that lie below and above the threshold $\pi/2$. The orange curves are the transport lines.

and eliminating r in the above integral via $r = (s \cos x)/((1-s) \sin x)$ we obtain the identity

$$\frac{d}{ds} \int_{[0, \pi/2]} \psi(x) d\mu_s(x) = \int_{[0, \pi/2]} \left(4\xi(s, x)\psi(x) + \Xi(s, x)\psi'(x) \right) d\mu_s(x).$$

Since ξ satisfies the Hamilton–Jacobi equation $\frac{d}{ds}\xi + \frac{1}{2}|\nabla\xi|^2 + 2\xi^2 = 0$, we conclude that the pair (μ, ξ) indeed satisfies the equation (5.1).

It is interesting to note that ξ is independent of the measure $\mathbf{n}_0 \widehat{\gamma}$. All the information about the precise form of the connecting geodesic is solely encoded in the information how the singularity for $s \searrow 0$ and $s \nearrow 1$ are formed, and this information is exactly contained in $\mathbf{n}_0 \widehat{\gamma}$.

5.3 Dilation of measures

For the Kantorovich–Wasserstein distance there is a geodesic connection between any measure μ_1 and the Dirac measure $\mu_0 = \mu_1(\Omega)\delta_{y_0}$ by radially dilating the measure, viz.

$$\mu^{\text{KaWa}}(s) = X^{\text{KaWa}}(s, \cdot)_{\#} \mu_1 \quad \text{where } X^{\text{KaWa}}(s, x) = (1-s)y_0 + sx.$$

This dilation corresponds to the solution $\xi(s, x) = \frac{1}{2}|x - y_0|^2/s$ of the standard Hamilton–Jacobi equation $\frac{d}{ds}\xi + \frac{1}{2}|\nabla\xi|^2 = 0$, and $\frac{d}{ds}\mu(s) + \text{div}(\mu\nabla\xi) = 0$.

A possible generalization of this dilation to the Hellinger–Kantorovich distance is given by a solution ξ of the modified Hamilton–Jacobi equation $\frac{d}{ds}\xi + \frac{1}{2}|\nabla\xi|^2 + 2\xi^2 = 0$ having the form

$$\xi(s, x) = \frac{\zeta(x)}{2s} \quad \text{with } |\nabla\zeta(x)|^2 + 4\zeta(x)^2 - 4\zeta(x) \equiv 0.$$

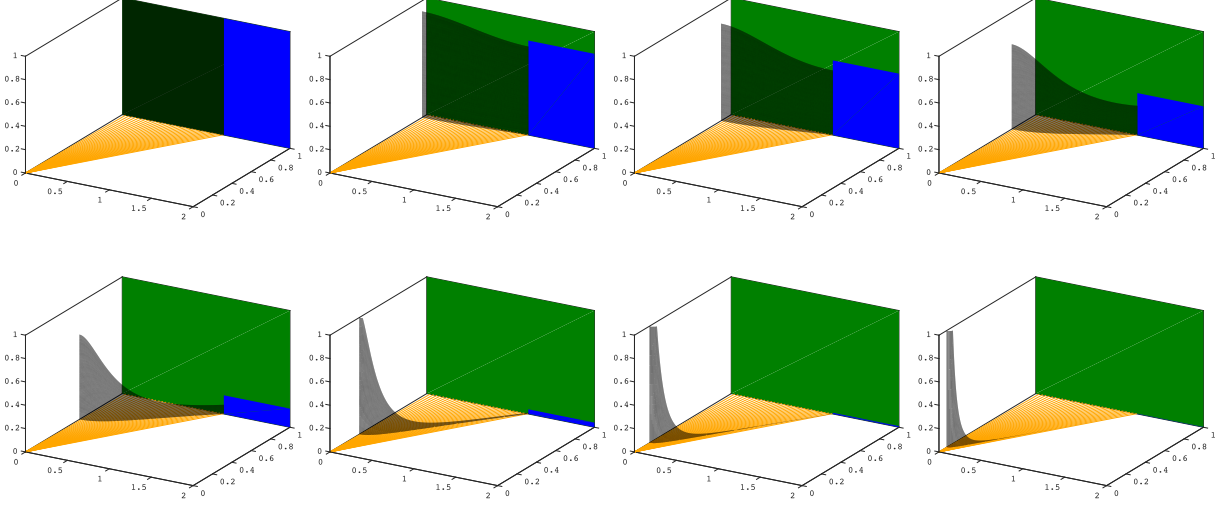


Figure 6: Geodesic curve $s \mapsto \mu(s)$ connecting line measure and single Dirac measure in Example 5.4(ii) for different values of s . Here green denotes the measure μ_1 while the blue and the black parts correspond to the parts of $\mu(s)$ that lie below and above the threshold $\pi/2$. The orange curves are the transport lines.

The trivial solutions $\zeta \equiv 0$ and $\zeta \equiv 1$ correspond to constant and pure Hellinger geodesics, respectively. However, there are many other solutions, e.g.

$$\tilde{\zeta}(x) = \begin{cases} (\sin(|x|))^2 & \text{for } |x| \leq \pi/2, \\ 1 & \text{for } |x| \geq \pi/2; \end{cases} \quad \text{and } \bar{\zeta}(x) = \min\{\tilde{\zeta}(x^k - y_0^k) \mid k = 1, \dots, d\}$$

if $|y_0^j - y_0^k| \geq \pi$ for all $j \neq k$.

Staying with $\zeta = \tilde{\zeta}$ we see that an arbitrary measure μ_1 can be connected to $\mu_0 = a_0\delta_0$, with $a_0 > 0$ fixed, by the geodesic connection $\mu(s) = \mu_s$ given via

$$\begin{aligned} \int_{\Omega} \psi(x) d\mu_s(x) &= \int_{\Omega \cap \{|x| \geq \pi/2\}} s^2 \psi(x) d\mu_1(x) \\ &+ \int_{\Omega \cap \{|x| < \pi/2\}} ((1-s^2)(\cos|x|)^2 + s^2) \psi\left(\arctan[s \tan|x|] \frac{x}{|x|}\right) d\mu_1(x), \end{aligned}$$

where the first term on the right-hand side denotes the pure Hellinger part, while the second term involves the concentration into $a_0\delta_0$ for $s \searrow 0$, where the total mass at $s = 0$ equals $a_0 := \int_{\{|x| < \pi/2\}} (\cos|x|)^2 d\mu_1$.

Again it is easy to show that the pair (μ, ξ) with $\xi(s, x) = \tilde{\zeta}(x)/(2s)$ satisfies the formal equation (5.1) for geodesic curves. We also note that the dilation operation is unique, even if μ_1 has positive mass on the sphere $\{|x| = \pi/2\}$. This is because of the fixed function ξ . Of course there might be other geodesic curves connecting $\mu_0 = a_0\delta_0$ and μ_1 , e.g. for $\mu_1 = a_1\delta_{y_1}$ with $|y_1| = \pi/2$, where we have all the solutions constructed in Section 5.2.

Example 5.4 (i) As a more concrete example, we consider in $\Omega = [0, 1] \times [0, 2]$ the measures $\mu_1 = \sum_{k=1}^N \delta_{x_k}$, $N \geq 2$, and $\mu_0 = a_0\delta_0$, where

$$x_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{k-1}{N-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \text{for } k = 1, \dots, N, \quad \text{and} \quad a_0 = \sum_{k: |x_k| < \pi/2} \cos(|x_k|)^2.$$

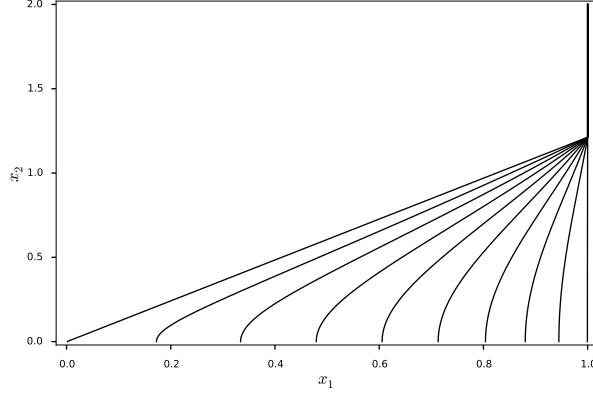


Figure 7: Supports of geodesic curve $\mu(s)$ in Example 5.4(ii) for different values of s .

Using the formula above, we obtain the geodesic connection

$$\mu(s) = \sum_{k: |x_k| \geq \pi/2} s^2 \delta_{x_k} + \sum_{k: |x_k| < \pi/2} ((1-s^2)(\cos |x_k|)^2 + s^2) \delta_{\rho_k(s)x_k},$$

$$\text{where } \rho_k(s) = \frac{1}{|x_k|} \arctan [s \tan |x_k|].$$

The geodesic connection $\mu(s)$ is depicted in Figure 5 for different values of s .

(ii) Similarly, we can compute a geodesic connection $\mu(s)$ for the line measure $\mu_1 = \delta_1 \otimes \mathcal{L}^1|_{[0,2]}$ which is collapsed into the measure $\mu_0 = a_0 \delta_0$ with $a_0 = \int_0^2 (\cos(y))^2 dy$. In this case, $\mu(s)$ is concentrated on the set given by the function

$$X(s; x) = \begin{cases} \rho(s; x)x, & \text{for } |x| \leq \pi/2, \\ x & \text{otherwise,} \end{cases} \quad \text{for } s \in [0, 1] \text{ and } x \in \text{supp } \mu_1,$$

where $\rho(s; x) = \arctan(s \tan |x|)/|x|$. On these curves the density with respect to the one-dimensional Hausdorff measure is for $y \in \Omega$ and $\tilde{X}_s(x_2) := X(s; (1, x_2))$ for $x_2 \in [0, 2]$, given by

$$\tilde{a}(s, y) = \frac{a(s; \cdot)}{|\partial_{x_2} X(s; \cdot)|} \circ X_s^{-1}(y)$$

where the profile a reads

$$a(s; x) = \begin{cases} (1-s^2)(\cos |x|)^2 + s^2 & \text{for } |x| \leq \pi/2, \\ s^2 & \text{otherwise.} \end{cases}$$

The curve $\mu(s)$ is shown in Figure 6.

5.4 Transport of characteristic functions

Here, we discuss a method to explicitly construct the geodesic connection between two characteristic functions $\mu_j = a_j \chi_{[x_j^{\text{left}}, x_j^{\text{right}}]}$ where $\Omega \subset \mathbb{R}^1$. However, to simplify notations we will restrict to the specific case

$$\mu_0 = \chi_{[-\pi/4, \pi/4]} dx \quad \text{and} \quad \mu_1 = \chi_{[\pi/2, \pi]} dx.$$

Obviously, we have the Hellinger parts $\mu_0^\perp = \chi_{[-\pi/4, 0]} dx$ and $\mu_1 = \chi_{[3\pi/4, \pi]} dx$, which are absorbed and generated, respectively, without any interaction with the transport in between.

To construct a transport geodesic from $\mu_0^{\text{tr}} = \chi_{I_0} dx$ and $\mu_1^{\text{tr}} = \chi_{I_1} dx$, with $I_0 =]0, \pi/2[$ and $I_1 =]\pi/2, 3\pi/4[$, we find the functions $r_j : I_j \rightarrow \mathbb{R}$ via minimizing the entropy-transport functional $\mathcal{ET}(\eta; \mu_0, \mu_1)$. We establish the calibration measure η via a map $h : I_0 \rightarrow I_1$ in the form

$$\int_{I_0 \times I_1} \Psi(x_0, x_1) d\eta(x_0, x_1) = \int_{I_0} \Psi(x, h(x)) f(x) dx.$$

Checking the marginal conditions $\Pi_{\#}^j \eta = \varrho_j dx$ we find $f(x) = \varrho_0(x) = \varrho_1(h(x))h'(x)$. Moreover, the optimality conditions in Theorem 3.7 give, for all $x \in I_0$ and $y \in I_1$,

$$\varrho_0(x)\varrho_1(h(x)) = [\cos(h(x)-x)]^2 \quad \text{and} \quad \varrho_0(x)\varrho_1(y) \geq [\cos(y-x)]^2. \quad (5.7)$$

Deriving the first-order optimality conditions at $y = h(x)$ from the second relation in (5.7) we find

$$2 \sin(h(x)-x) \cos(h(x)-x) h'(x) = \varrho_0'(x) \varrho_1(h(x)) h'(x) = \varrho_0(x) \varrho_0'(x) = \frac{1}{2} (\varrho_0(x)^2)'$$

Since the first relation in (5.7) has the form $\varrho_0(x)^2 = h'(x) [\cos(h(x)-x)]^2$ we find

$$h''(x) = 2(h'(x)^2 + h'(x)) \tan(h(x)-x), \quad h(0) = \pi/2, \quad h(\pi/4) = 3\pi/4,$$

which has a unique monotone solution. Indeed, to see this let $h(x) = \pi/2 + x - w(x)$, where now $w(0) = w(\pi/4) = 0$ and $w > 0$. Then the ODE reads $w'' = b(w')c(w)$ for suitable functions b and c . Rewriting it in the form $w'w''/b(w') = c(w)w'$ we find $A(w(x)) = C(w(x)) + \gamma$, where $B'(y) = y/b(y)$ and $C'(y) = c(y)$. An explicit calculation and exponentiating both sides yields

$$\frac{\sqrt{1-w'(x)}}{2-w'(x)} = c_* \sin w(t), \quad w(0) = w(\pi/4) = 0.$$

Solving for w' we find $w' = g_{\pm}(c_* \sin w)$ with $g_{\pm}(a) = (4a^2 - 1 \pm \sqrt{1-4a^2})/(2a^2)$ for $0 < a \leq 1/2$, where $\pm g_{\pm}(a) > 0$ for $a \neq 1/2$ and $g_{\pm}(1/2) = 0$. Thus, w will have a unique maximum $w_* = w(w_*)$ with $c_* \sin w_* = 1/2$, and w_* can be determined uniquely from

$$\begin{aligned} \frac{\pi}{4} &= \int_0^{x_*} dx + \int_{x_*}^{\pi/4} dx = \int_0^{w_*} \frac{dw}{g_+(w)} + \int_0^{w_*} \frac{dw}{-g_-(w)} \\ &= \int_0^{w_*} \frac{\sin w_*}{\sqrt{(\sin w_*)^2 - (\sin w)^2}} dw = K(w_*) \sin w_*, \end{aligned}$$

where K is the elliptic K function. Numerically, we find $w_* = 0.4895$ and thus $c_* = 1.0634$.

Now it is straightforward to show that there is exactly one $c_* > 0$ such that a solution with $w(x) > 0$ and $w'(x) < 1$ exists.

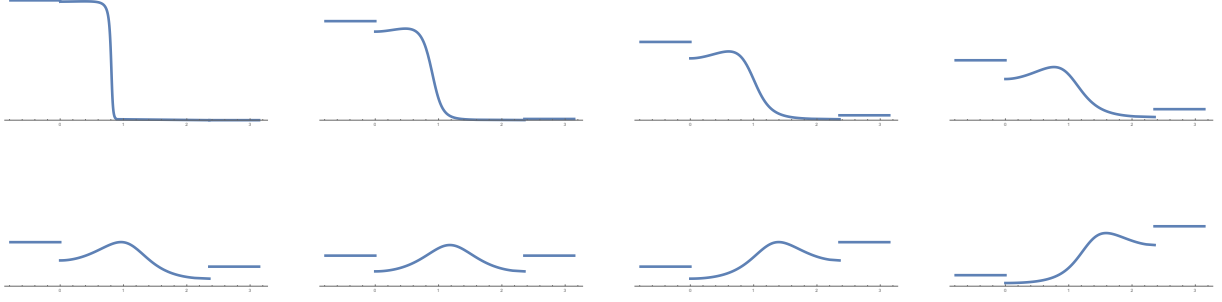


Figure 8: The densities $f(s, \cdot)$ of the geodesic curves $\mu(s) = f(s, \cdot) dx$ connecting $\mu_0 = \chi_{[-\pi/4, \pi/4]} dx$ and $\mu_1 = \chi_{[\pi/2, \pi]} dx$ for $s = 0.01, 0.1, 0.2, \dots, 0.7$.

Based on the function h the densities ϱ_0 and ϱ_1 are explicitly known and we may write the geodesic connection $\mu_s = \mu(s)$ for $\mu_0^{\text{tr}} = \chi_{I_0} dx$ and $\mu_1^{\text{tr}} = \chi_{I_1} dx$ as in (3.21)

$$\begin{aligned} \int_{[0, 3\pi/4]} \psi(y) d\mu_s(y) &= \int_{I_0} \widehat{R}(s, x)^2 \psi(Y(s, x)) \varrho_0(x) dx \\ \text{with } \overline{R}(s, x)^2 &= (1-s)^2 r_0^2(x) + s^2 r_1^2(h(x)) + 2s(1-s) \\ \text{and } Y(s, y) &= x + \arccos\left(\frac{(1-s)r_0(x) + sr_1(h(x)) \cos(h(x)-x)}{\overline{R}(s, x)}\right), \end{aligned}$$

where $r_j(x_j) = (\varrho_j(x_j))^{-1/2}$. Thus, the density $f(s, \cdot)$ of μ_s satisfies the relation

$$\begin{aligned} \int_0^{Y(s, x)} f(s, y) dy &= \int_0^{3\pi/4} \chi_{\{y \leq Y(s, x)\}}(y) d\mu_s(y) \\ &= \int_{I_0} \overline{R}(s, x_0)^2 \chi_{\{y \leq Y(s, x)\}}(Y(s, x_0)) \varrho_0(x_0) dx_0 = \int_0^x \overline{R}(s, x_0)^2 \varrho_0(x_0) dx_0. \end{aligned}$$

Differentiation with respect to x gives the explicit formula

$$f(s, Y(s, x)) = \frac{\overline{R}(s, x)^2 \varrho_0(x)}{\partial_x Y(s, x)}.$$

In Figure 8, we plot the densities together with the corresponding Hellinger parts.

5.5 Towards a characterization of all geodesic connecting two measures

Here we discuss the question of describing all geodesic curves for two given measures μ_0 and μ_1 . As we have seen in Section 5.2, the set of all these curves can be very big, in fact even infinite dimensional. The final aim would be to define a geometric tangent cone in the sense of [AGS05, Ch. 12].

The major tool in understanding the structure of all geodesic connections is Corollary 4.4, which states that all geodesic curves $s \mapsto \mu(s)$ are given as projections $\mu(s) = \mathfrak{P}\lambda(s)$

of geodesic curves λ in $\mathcal{M}_2(\mathfrak{C}_\Omega)$. Thus, writing (3.16) more explicitly via

$$\mathbf{HK}(\mu_0, \mu_1) = \min \left\{ \iint_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} \mathbf{d}_{\mathfrak{C}}(z_0, z_1)^2 d\gamma(z_0, z_1) \mid \mathfrak{P}\Pi_{\#}^0 \gamma = \mu_0, \mathfrak{P}\Pi_{\#}^1 \gamma = \mu_1 \right\}, \quad (5.8)$$

we can define the set of optimal plans via

$$\text{Opt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1) := \left\{ \gamma \in \mathcal{M}_2(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega) \mid \gamma \text{ is optimal in (5.8)} \right\},$$

which is a convex set. Every optimal plan γ gives rise to different geodesics $\lambda(s) = Z(s; \cdot, \cdot)_{\#} \gamma$ in $\mathcal{M}_2(\mathfrak{C}_\Omega)$. While for different γ the geodesics λ are different, the same is no longer true for the projections $\mu(\cdot) = \mathfrak{P}\lambda(\cdot)$.

The major redundancies in the set $\text{Opt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1)$ of optimal plans is seen through the scaling invariance given in relation (3.18). If γ contains a transport of mass m along a cone geodesic connecting $[x_0, r_0]$ and $[x_1, r_1]$, then the contribution to the projection $\mu(s) = \mathfrak{P}\lambda(s)$ is equal to a transport of mass $\vartheta^2 m$ along the cone geodesic connecting $[x_0, r_0/\vartheta]$ and $[x_1, r_1/\vartheta]$ for all $\vartheta > 0$. Thus, we can define a normalization operator N action on plans γ , that does not change the projection.

For this we consider the partition of $\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega$ given by the sets \mathfrak{G} , \mathfrak{G}'_{12} , \mathfrak{G}'_1 , \mathfrak{G}'_2 , and \mathfrak{G}'_0 :

$$\begin{aligned} \mathfrak{G} &:= \left\{ ([x_1, r_1], [x_2, r_2]) \in \mathfrak{C}_\Omega \times \mathfrak{C}_\Omega : r_1 r_2 > 0, |x_1 - x_2| < \pi/2 \right\}, \\ \mathfrak{G}'_{12} &:= \left\{ ([x_1, r_1], [x_2, r_2]) \in \mathfrak{C}_\Omega \times \mathfrak{C}_\Omega : r_1 r_2 > 0, |x_1 - x_2| \geq \pi/2 \right\}, \\ \mathfrak{G}'_1 &:= \left\{ ([x_1, r_1], \mathfrak{o}) : r_1 > 0 \right\}, \quad \mathfrak{G}'_2 := \left\{ (\mathfrak{o}, [x_2, r_2]) : r_2 > 0 \right\}, \quad \mathfrak{G}'_0 := \{\mathfrak{o}\} \times \{\mathfrak{o}\}. \end{aligned}$$

With these sets we define the scaling function

$$\vartheta(z_1, z_2) := \begin{cases} \left(r_1 r_2 \cos(|x_1 - x_2|) \right)^{1/2} & \text{if } (z_1, z_2) \in \mathfrak{G}, \\ r_1 & \text{if } (z_1, z_2) \in \mathfrak{G}'_{1,2} \cup \mathfrak{G}'_1, \\ r_2 & \text{if } (z_1, z_2) \in \mathfrak{G}'_2, \\ 1 & \text{if } z_1 = z_2 = \mathfrak{o} \end{cases}$$

and employ the dilation map h_ϑ from (3.17) to generate the corresponding rescaling $N : \mathcal{M}_2(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega) \rightarrow \mathcal{M}_2(\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega)$ by $N\gamma := (h_\vartheta)_\#(\vartheta^2 \gamma) \llcorner (\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega \setminus \mathfrak{G}'_0)$, where \llcorner denotes restriction of a measure. By the definition of \mathfrak{P} and the scaling property of $\mathbf{d}_{\mathfrak{C}}$ we first find $\mathfrak{P}\Pi_{\#}^j \gamma = \mathfrak{P}\Pi_{\#}^j N\gamma$ and $\iint_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} \mathbf{d}_{\mathfrak{C}}(z_0, z_1)^2 d\gamma = \iint_{\mathfrak{C}_\Omega \times \mathfrak{C}_\Omega} \mathbf{d}_{\mathfrak{C}}(z_0, z_1)^2 d(N\gamma)$. Hence, for each $\gamma \in \text{Opt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1)$ we again have $N\gamma \in \text{Opt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1)$. Thus, we define the normalized optimal plans via

$$\text{NormOpt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1) := \left\{ \gamma \in \text{Opt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1) \mid \gamma = N\gamma \right\}, \quad (5.9)$$

which is a much smaller but still closed and convex set.

Using the scaling properties of the geodesics on \mathfrak{C}_Ω , which are given via the interpolating functions $Z = (X, R)$ as $\mu(s) = \mathfrak{P}Z(s; \cdot, \cdot)_{\#} \gamma$ (cf. (3.21)), and the fact that ϑ depends on x_1, x_2 only through their distance $|x_1 - x_2|$, we also see that γ and $N\gamma$ generate the same geodesic, viz.

$$\hat{\mu}_\gamma(s) := \mathfrak{P}(Z(s; \cdot, \cdot)_{\#} \gamma) = \mathfrak{P}(Z(s; \cdot, \cdot)_{\#} (N\gamma)) = \hat{\mu}_{N\gamma}(s).$$

With these preparations we are able to prove that there exists a unique geodesic connecting μ_0 to μ_1 if μ_0 is absolutely continuous with respect to the Lebesgue measure. More precisely, we show that in this case N eliminates all redundancies: $\text{NormOpt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1)$ contains only one element and $N\gamma$ characterize the geodesic connection of μ_0 and μ_1 uniquely.

Theorem 5.5 *For every couple μ_0, μ_1 in $\mathcal{M}(\Omega)$ with μ_0 absolutely continuous w.r.t. the Lebesgue measure, there exists a unique geodesic μ connecting μ_0 to μ_1 and a unique $\gamma \in \text{NormOpt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1)$. In particular, for each geodesic curve μ connecting μ_0 to μ_1 there exists a unique $\gamma \in \text{NormOpt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1)$ such that $\mu = \hat{\mu}_\gamma$.*

Proof: By the above discussion we have just to check the uniqueness of γ . We first show that any $\gamma \in \text{NormOpt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1)$ does not charge \mathfrak{G}'_{12} .

Since γ is optimal, the rescaling given by N and [LMS15, Thm. 7.21] yield that $\gamma(\mathfrak{G}'_{12}) = \gamma(\tilde{\mathfrak{G}}'_{12})$ where

$$\tilde{\mathfrak{G}}'_{12} = \left\{ ([x_1, 1], [x_2, r_2]) \in \mathfrak{C}_\Omega \times \mathfrak{C}_\Omega : r_2 > 0, |x_1 - x_2| = \pi/2 \right\} \subset \mathfrak{G}'_{12}.$$

Setting $\tilde{\mu}_0 := \Pi_{\sharp}^1(\gamma \llcorner \tilde{\mathfrak{G}}'_{12}) = \mathfrak{P}\Pi_{\sharp}^1(\gamma \llcorner \mathfrak{G}'_{12})$, we have $\tilde{\mu}_0 = \eta\mu_0 \leq \mu_0$; in particular $\tilde{\mu}_0$ is absolutely continuous with respect to the Lebesgue measure. If we set $f(x) := \min_{y \in \text{supp}(\mu_1)} |x - y|$, applying [LMS15, Thm. 6.3(b)] we deduce that

$$f(x) = \pi/2 \quad \text{for } \tilde{\mu}_0\text{-a.e. } x \in \Omega.$$

Applying the co-area formula to f , see [Fed69, Lem. 3.2.34], we have

$$\mathcal{L}^d \left(\left\{ x \in \mathbb{R}^d \mid \pi/2 - \varepsilon \leq f(x) \leq \pi/2 + \varepsilon \right\} \right) = \int_{\pi/2 - \varepsilon}^{\pi/2 + \varepsilon} \mathcal{H}^{d-1}(f^{-1}(p)) dp \quad \text{for every } \varepsilon > 0,$$

so that passing to the limit as $\varepsilon \downarrow 0$ we get $\mathcal{L}^d(\{x \in \mathbb{R}^d \mid f(x) = \pi/2\}) = 0$. It follows that $\tilde{\mu}_0$ is the null measure and $\gamma(\mathfrak{G}'_{12}) = 0$.

Let us now suppose that $\gamma', \gamma'' \in \text{NormOpt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1)$.

Combining Theorems 7.2.1 (iv) and 6.7 of [LMS15] (where we use the absolute continuity of μ_0 again) we see that the restrictions of γ' and γ'' to \mathfrak{G} coincide. By subtracting this common part from both of them and the corresponding homogeneous marginals from μ_0 and μ_1 , it is not restrictive to assume that γ' and γ'' are concentrated in the complement of \mathfrak{G} . By the previous claim, we obtain that γ' and γ'' are concentrated on $\mathfrak{G}'_1 \cup \mathfrak{G}'_2$.

It is also easy to see that the restrictions of γ' and γ'' to \mathfrak{G}_i , $i = 1, 2$, coincide as well: considering e.g. \mathfrak{G}'_1 , by construction we have that $\Pi_{\sharp}^1\gamma' = \Pi_{\sharp}^1\gamma'' = \mu_1 \otimes \delta_1$, whereas $\Pi_{\sharp}^2\gamma' = \Pi_{\sharp}^2\gamma'' = \mu_1(\mathbb{R}^d)\delta_0$. It follows that $\gamma' \llcorner \mathfrak{G}'_1 = \gamma'' \llcorner \mathfrak{G}'_1 = (\mu_1 \otimes \delta_1) \otimes \delta_0$. A similar argument holds for \mathfrak{G}'_2 . This proves the result. \blacksquare

The major point in the above proof is to show that $\gamma(\tilde{\mathfrak{G}}'_{12}) = 0$, i.e. there is no transport over the distance $\pi/2$. From Section 5.2 we know that in the opposite case $\text{NormOpt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1)$ can be infinite dimensional.

We expect that, by refining the arguments above and using the dual characterization of \mathbf{HK} , it is possible to prove that for each geodesic curve connecting μ_0 and μ_1 (both not necessarily absolutely continuous w.r.t. \mathcal{L}^d) there exists a unique $\gamma \in \text{NormOpt}_{\mathbf{HK}}^{\mathfrak{C}}(\mu_0, \mu_1)$

such that $\mu = \hat{\mu}_\gamma$. As in the Kantorovich-Wasserstein case, the optimal plan γ should be uniquely determined by fixing an intermediate point $\hat{\mu}_\gamma(s)$ with $s \in]0, 1[$, along a geodesic. In particular, geodesics for the Hellinger-Kantorovich distance \mathbf{HK} in $\mathcal{M}(\Omega)$ will then be nonbranching. Indeed, for the rich set of geodesics connecting two Dirac masses at distance $\pi/2$ discussed in Section 5.2 this can be shown by direct inspection. We will address these questions in a forthcoming paper.

5.6 \mathbf{HK} is not semiconcave

The metric on a geodesic space (Y, \mathbf{d}) is called K -semiconcave, if for all points $y_0, y_1, y_* \in Y$ and all minimal geodesic curves $\tilde{y} : [0, 1] \rightarrow Y$ with $\tilde{y}(i) = y_i$ for $i \in \{0, 1\}$, we have

$$\mathbf{d}(\tilde{y}(s), y_*)^2 \geq (1-s)\mathbf{d}(y_0, y_*)^2 + s\mathbf{d}(y_1, y_*)^2 - Ks(1-s)\mathbf{d}(y_0, y_1)^2. \quad (5.10)$$

It is well-known that the Wasserstein distance on a domain $\Omega \subset \mathbb{R}^d$ is 1-semiconcave (such that $(\mathcal{P}_2(\Omega), \mathbf{W})$ is a positively curved (PC) space, see [AGS05, Ch. 12.3]), and it is easy to check that the Hellinger-Kakutani distance \mathbf{H} is 1-semiconcave. Indeed, with the notation of Section 2.3 we have the identity

$$\mathbf{H}(\mu^{\mathbf{H}}(s), \mu_*)^2 = (1-s)\mathbf{H}(\mu_0, \mu_*)^2 + s\mathbf{H}(\mu_1, \mu_*)^2 - s(1-s)\mathbf{H}(\mu_0, \mu_1)^2$$

for any $\mu_0, \mu_1, \mu_* \in \mathcal{M}(\Omega)$, where $s \mapsto \mu^{\mathbf{H}}(s) \in \mathcal{M}(\Omega)$ is the Hellinger geodesic from (2.8).

In contrast, the Hellinger-Kantorovich distance is not K -semiconcave for any K if Ω is not one-dimensional. For the one-dimensional case $\Omega \subset \mathbb{R}$ it is shown in [LMS15, Thm. 8.9] that $(\mathcal{M}([a, b]), \mathbf{HK})$ is a PC space, which means that 1-semiconcavity holds. The following result shows that $(\mathcal{M}(\Omega), \mathbf{HK})$ is not a PC space if Ω has dimension $d \geq 2$.

For this case we consider a simple example, namely

$$\mu_0 = \delta_{x_0}, \quad \mu_1 = \delta_{x_1}, \quad \mu_* = b\delta_z, \quad \text{with } x_0 = 0, \quad x_1 = \frac{\pi}{2}e_1, \quad z = \frac{\pi}{4}e_1 + ye_2,$$

where e_1 and e_2 are the first two unit vectors and $y > 0$. As a geodesic curve we choose

$$\mu(s) = a(s)\delta_{\rho(s)e_1} \quad \text{with } a(s) = (1-s)^2 + s^2 \text{ and } \rho(s) = \arctan(s/(1-s)).$$

We have $\mathbf{HK}(\mu_0, \mu_1)^2 = 2$ and $\mu(1/2) = \frac{\pi}{4}e_1$, and all the quantities in the semiconcavity condition (5.10) can be evaluated explicitly. This yields a lower bound for K , namely

$$\begin{aligned} K &\geq \frac{\frac{1}{2}\mathbf{HK}(\mu_0, \mu_*)^2 + \frac{1}{2}\mathbf{HK}(\mu_1, \mu_*)^2 - \mathbf{HK}(\mu(\frac{1}{2}), \mu_*)^2}{\frac{1}{4}\mathbf{HK}(\mu_0, \mu_1)^2} \\ &= 1 + \sqrt{b} \phi(y) \quad \text{with } \phi(y) = 1 + \sqrt{8} \cos_{\pi/2}(y) - 4 \cos_{\pi/2} \sqrt{y^2 + \pi^2/16}, \end{aligned}$$

where $\cos_{\pi/2} a = \cos(\min\{|a|, \pi/2\})$. Since $\phi(y) > 0$ for $y \in]0, \sqrt{3}\pi/4[$ and since b can be chosen arbitrarily large, we see that there cannot exist a finite K such that \mathbf{HK} is K -semiconcave.

Acknowledgments

M.L. was partially supported by the Einstein Stiftung Berlin via the ECMath/MATHEON project SE2. A.M. was partially supported by DFG via project C5 within CRC 1114 (Scaling cascades in complex systems) and by the ERC AdG. 267802 *AnaMultiScale*. G.S. was partially supported by PRIN10/11 grant from MIUR for the project *Calculus of Variations*.

References

- [AD*11] S. ADAMS, N. DIRR, M. A. PELETIER, and J. ZIMMER. From a large-deviations principle to the Wasserstein gradient flow: a new micro-macro passage. *Comm. Math. Phys.*, 307(3), 791–815, 2011.
- [AGS05] L. AMBROSIO, N. GIGLI, and G. SAVARÉ. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.
- [BBI01] D. BURAGO, Y. BURAGO, and S. IVANOV. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.
- [BeB00] J.-D. BENAMOU and Y. BRENIER. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numer. Math.*, 84(3), 375–393, 2000.
- [CP*15a] L. CHIZAT, G. PEYRÉ, B. SCHMITZER, and F.-X. VIALARD. An interpolating distance between optimal transport and Fisher–Rao. *arXiv:1506.06430v2*, 2015.
- [CP*15b] L. CHIZAT, G. PEYRÉ, B. SCHMITZER, and F.-X. VIALARD. Unbalanced optimal transport: geometry and Kantorovich formulation. *arXiv:1508.05216v1*, 2015.
- [DaS08] S. DANERI and G. SAVARÉ. Eulerian calculus for the displacement convexity in the Wasserstein distance. *SIAM J. Math. Analysis*, 40, 1104–1122, 2008.
- [Fed69] H. FEDERER. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
- [Hel09] E. HELLINGER. Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen (in German). *J. Reine Angew. Math.*, 136, 210–271, 1909.
- [JKO97] R. JORDAN, D. KINDERLEHRER, and F. OTTO. Free energy and the fokker-planck equation. *Physica D*, 107(2-4), 265–271, 1997.
- [JKO98] R. JORDAN, D. KINDERLEHRER, and F. OTTO. The variational formulation of the Fokker-Planck equation. *SIAM J. Math. Analysis*, 29(1), 1–17, 1998.
- [Kak48] S. KAKUTANI. On equivalence of infinite product measures. *Annals Math.*, 49(1), 214–224, 1948.
- [KMV15] S. KONDRATYEV, L. MONSAINGEON, and D. VOROTNIKOV. A new optimal transport distance on the space of finite radon measures. *arXiv 1505.07746v2*, 2015.
- [LiM13] M. LIERO and A. MIELKE. Gradient structures and geodesic convexity for reaction-diffusion systems. *Phil. Trans. Royal Soc. A*, 371(2005), 20120346, 28, 2013.
- [Lis07] S. LISINI. Characterization of absolutely continuous curves in Wasserstein spaces. *Calc. Var. Part. Diff. Eqns.*, 28(1), 85–120, 2007.
- [LMS15] M. LIERO, A. MIELKE, and G. SAVARÉ. Optimal entropy-transport problems and the Hellinger–Kantorovich distance. *Preprint arXiv*, August 2015.
- [Man07] S. MANIGLIA. Probabilistic representation and uniqueness results for measure-valued solutions of transport equations. *J. Math. Pures Appl. (9)*, 87(6), 601–626, 2007.
- [Mie11] A. MIELKE. A gradient structure for reaction-diffusion systems and for energy-drift-diffusion systems. *Nonlinearity*, 24, 1329–1346, 2011.
- [Mie13] A. MIELKE. Thermomechanical modeling of energy-reaction-diffusion systems, including bulk-interface interactions. *Discr. Cont. Dynam. Systems Ser. S*, 6(2), 479–499, 2013.
- [OnM53] L. ONSAGER and S. MACHLUP. Fluctuations and irreversible processes. *Phys. Rev.*, 91(6), 1505–1512, 1953.
- [Ons31] L. ONSAGER. Reciprocal relations in irreversible processes, I+II. *Physical Review*, 37, 405–426, 1931. (part II, 38:2265-2279).
- [Ott98] F. OTTO. Dynamics of labyrinthine pattern formation in magnetic fluids: a mean-field theory. *Arch. Rational Mech. Anal.*, 141(1), 63–103, 1998.
- [Ott01] F. OTTO. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26, 101–174, 2001.

- [OtW05] F. OTTO and M. WESTDICKENBERG. Eulerian calculus for the contraction in the Wasserstein distance. *SIAM J. Math. Analysis*, 37, 1227–1255, 2005.
- [PQV14] B. PERTHAME, F. QUIRÓS, and J. L. VÁZQUEZ. The Hele-Shaw asymptotics for mechanical models of tumor growth. *Arch. Rational Mech. Anal.*, 212(1), 93–127, 2014.
- [Sch96] A. SCHIED. Sample path large deviations for super-brownian motion. *Probability Theory and Related Fields*, 104(3), 319–347, 1996.
- [Vil09] C. VILLANI. *Optimal transport. Old and new*. Berlin: Springer, 2009.